CDT and Hořava-Lifshitz QG in Two

二维下的 CDT 与霍拉瓦-里夫希茨量子引力

Dimensions

维度

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摘要

Two-dimensional causal dynamical triangulations (2d CDT) is a lattice model of quantum geometry. In 2d CDT, one can deal with the quantum effects analytically and explore the physics through the continuum limit. The continuum theory is known to be two-dimensional projectable Hořava-Lifshitz quantum gravity (2d projectable HL QG). In this chapter, we wish to review the very relation between 2d CDT and 2d projectable HL QG in detail.

二维因果动态三角剖分 (2d CDT) 是量子几何的一种格点模型。在二维因果动态三角剖分中，人们可以解析处理量子效应，并通过连续极限探究相关物理规律。已知该连续理论就是二维可投影霍拉瓦-里夫希茨量子引力 (二维可投影 HL QG)。本章我们将详细回顾二维因果动态三角剖分与二维可投影霍拉瓦-里夫希茨量子引力之间的关联。

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格点量子引力——因果动力学三角剖分 (CDT)。霍拉瓦-里夫希茨量子引力——二维模型——虫洞

Introduction

引言

Two-dimensional toy models of quantum gravity are a very useful playground for understanding the quantum nature of geometry quantitatively. This is because many of them are simple enough to be dealt with analytically and complex enough to observe nontrivial quantum effects. Lattice regularizations in particular are known to be quite powerful tools for investigating non-perturbative quantum effects analytically. Among these, two-dimensional Euclidean dynamical triangulations (2d EDT) [1-6] (see a pedagogical textbook [7]) and two-dimensional causal dynamical triangulations (2d CDT) [9] (see a detailed review [8]) are good practical examples. The former and the latter, respectively, are Euclidean and Lorentzian lattice models based on Regge's discretization of geometries [10].

二维量子引力 toy 模型是定量理解几何量子性质非常实用的研究平台。这是因为这类模型大多足够简单，可进行解析处理，同时又足够复杂，能观测到非平庸量子效应。尤其格点正规化是研究非微扰量子效应十分强大的解析工具，其中二维欧几里得动力三角化 (2d EDT)[1-6](参见教材 [7]) 和二维因果动力三角化 (2d CDT)[9](参见综述 [8]) 就是很好的典型实例。二者分别是基于 Regge 几何离散化方案 [10] 的欧几里得与洛伦兹格点模型。

2d EDT discretizes Euclidean geometries by equilateral triangles and defines a regularized quantum amplitude as a sum over distinct triangulated geometries. Matrix models and combinatorics can be used for calculating such a statistical sum analytically (whenever possible). By virtue of analytic treatments, one can explicitly remove the regularization through the continuum limit to calculate physical observables. What is remarkable is that exactly the same value of observables can be reproduced from a genuine continuum field theory called the Liouville quantum gravity [11-14]. This means that 2d EDT serves as a well-defined regularization of the Liouville quantum gravity.

> 2d EDT 用等边三角形对欧几里得几何进行离散化，将正规化量子振幅定义为对所有不同三角化几何的求和。矩阵模型和组合数学可用来解析计算这类统计和 (条件允许时)。通过解析处理，我们可以在连续极限下显式去除正规化，计算物理可观测量。值得注意的是，真正的连续场理论——刘维尔量子引力——可以得到完全相同的可观测量数值 [11-14]，这说明 2d EDT 是刘维尔量子引力一个定义良好的正规化方案。

2d CDT is a Lorentzian lattice model of quantum geometry, which respects a global time foliation and prohibits the creation of so-called baby universes. One can calculate the sum over such Lorentzian triangulated geometries using simple combinatorics and take the continuum limit to remove the regularization. All these processes can be performed analytically at least for the plain model without coupling to a matter. It has been shown in Ref. [15] that the resulting continuum theory is known to be in the same universality class of projectable Hořava-Lifshitz quantum gravity (projectable HL QG) [16] in two dimensions, which is different from the Liouville quantum gravity (In fact, it has been shown that a direct lattice discretization of 2 d projectable HL gravity, which has a lattice action different from that of 2 d CDT and reproduces the same large-scale physics in the continuum limit [18].).

> 2d CDT 是量子几何的洛伦兹格点模型，它满足全局时间叶分结构，禁止所谓婴儿宇宙的产生。我们可以利用简单组合数学计算这类洛伦兹三角化几何的求和，并通过取连续极限去除正规化。至少对于未耦合物质的纯模型，所有这些过程都可以解析完成。已有文献 [15] 表明，得到的连续理论与二维可投影霍拉瓦-利夫希茨量子引力 (可投影 HL QG) 属于同一普适类 [16]，和刘维尔量子引力不同 (事实上已有结论证明: 对 2 d 可投影 HL 引力直接进行格点离散化得到的格点作用量与 2 d CDT 不同，但在连续极限下仍能重现相同的大尺度物理 [18]。)。

2d HL QG is a quantum field theory in two dimensions, which has a preferred foliation structure. This model is invariant only under the subclass of diffeomorphisms that respects the foliation, known as the foliation-preserving diffeomorphisms (At the cost of full diffeomorphism invariance, HL QG has been designed originally as a model of quantum gravity in higher dimensions such that it has a good convergence at UV in keeping with unitarity and would approximately recover the diffeomorphism invariance at IR [16].). 2d projectable HL QG is a certain version of HL QG where the time-time component of the metric called the lapse function is projectable, i.e., a function only of time. In this chapter, we wish to explain in detail the relation between projectable HL QG and CDT in two dimensions (The relation between HL QG and CDT in four dimension has been pointed out first in Ref. [19] by looking at an observable called the spectral dimension, and the resemblance of the CDT phase diagram to a Lifshitz phase diagram has been shown in Ref. [20].).

> 二维 HL QG 是二维量子场论，具有优先叶分结构。该模型仅在保持叶分结构的微分同胚子群下不变，这类微分同胚称为保叶分微分同胚 (HL QG 最初设计时牺牲了完整微分同胚不变性，目的是得到高维量子引力模型，使其满足幺正性且在紫外区具有良好收敛性，并在红外区近似恢复微分同胚不变性 [16]。)。二维可投影 HL QG 是 HL QG 的一个特定版本: 该模型中称为时移函数的度规时间-时间分量是可投影的，即它仅是时间的函数。本章我们将详细解释二维可投影 HL QG 与 CDT 之间的关系 (四维下 HL QG 与 CDT 的关系最早由文献 [19] 通过谱维这个可观测量指出，CDT 相图与利夫希茨相图的相似性也已在文献 [20] 中给出。)。

In fact one can generalize 2d CDT in such a way that the creation and annihilation of (a finite number of) baby universes and the formation of wormholes (handles) are allowed to occur in keeping with the foliation structure. This model is called the generalized CDT (GCDT) introduced first as a continuum theory [21,22] and later defined at the discrete level [23, 24] . The full continuum description of GCDT is given by the so-called string field theory for CDT [22] in which the string means the one-dimensional closed spatial universe, and the baby universes and wormholes can be realized in terms of the splitting and joining interactions of strings.

事实上我们可以推广二维 CDT，在保持叶分结构的前提下允许 (有限个) 婴儿宇宙的产生和湮灭，也允许虫洞 (柄) 的形成。这个模型称为广义 CDT(GCDT)，它最初作为连续理论提出 [21,22]，之后在离散层面给出了定义 [23, 24] 。GCDT 的完整连续描述由所谓 CDT 弦场理论给出 [22]，该理论中弦指的是一维闭合空间宇宙，婴儿宇宙和虫洞可以通过弦的分裂和融合相互作用实现。

One of remarkable facts is that focusing on a certain amplitude, i.e., loop-to-loop amplitude, one can read off a one-dimensional effective theory that takes in all possible baby universe and wormhole contributions in an effective manner [25,26]: The 1d effective theory is a one-body quantum theory even though GCDT is a many-body theory that allows both creation and annihilation of strings. It is known that one can correctly reproduce the 1d effective theory if quantizing the projectable HL gravity with a certain bi-local interaction term [28, 29] . This topic will be treated in this chapter.

一个值得注意的结论是: 聚焦于特定振幅——即圈到圈振幅——我们可以推导出一个一维有效理论，它能以有效方式囊括所有可能的婴儿宇宙与虫洞贡献 [25,26]: 尽管 GCDT 是允许弦产生和湮灭的多体理论，该一维有效理论仍是单体量子理论。已知对带特定双局域相互作用项 [28, 29] 的可投影 HL 引力量子化，就能正确得到这个一维有效理论。本章节将讨论这一主题。

Furthermore, the 1d effective theory that includes all contributions of baby universes and wormholes is known to be reproduced if one assumes that the cosmological constant of the continuum limit of 2d CDT is not really a constant but fluctuates in time [30]. This idea leads to a certain realization of Coleman's mechanism about the cosmological constant [27] in the context of CDT [30], which will be also explained in this chapter.

此外，我们知道，如果假设二维 CDT 连续极限下的宇宙学常数并非真正的常数，而是随时间涨落 [30]，就能重现这个包含所有婴儿宇宙与虫洞贡献的一维有效理论。该想法为 CDT 框架下实现科尔曼关于宇宙学常数的机制 [27] 提供了一种方案 [30]，本章节也会对此进行讲解。

The rest of this chapter is organized as follows. In section "2d Causal Dynamical Triangulations," a self-contained introduction to 2d CDT is presented. We show that taking the continuum limit the physics of 2d CDT can be described as a one-dimensional quantum system with a Hamiltonian. 2d projectable HL QG is explained in section "2d Projectable Hořava-Lifshitz Quantum Gravity." Through the path-integral quantization, we read off the quantum Hamiltonian that is equivalent to the one obtained in the continuum limit of 2d CDT. Thereby one can confirm that the continuum limit of 2d CDT is 2d projectable HL QG. In section "Sum over All Wormholes and Baby Universes," we introduce GCDT that is a generalization of 2d CDT such that baby universes and wormholes are introduced so as to be compatible with the foliation and determine the 1d effective theory obtained through the sum over all genera. In particular, we explain in detail that quantizing 2d projectable HL gravity with a bi-local interaction yields the 1d effective theory and discuss Coleman's mechanism in 2d CDT. Section "Summary" is devoted to summary.

本章剩余部分结构安排如下: 在“二维因果动态三角剖分”小节，我们将完整介绍二维 CDT；我们证明，取连续极限后，二维 CDT 的物理可以用一个带哈密顿量的一维量子系统描述。在“二维可投影 Hořava-Lifshitz 量子引力”小节，我们讲解二维可投影 HL 量子引力；通过路径积分量子化，我们推导出的量子哈密顿量与二维 CDT 连续极限下得到的哈密顿量完全一致，由此可以确认二维 CDT 的连续极限就是二维可投影 HL 量子引力。在“所有虫洞与婴儿宇宙的求和”小节，我们引入 GCDT——它是二维 CDT 的推广，引入了与叶状结构相容的婴儿宇宙和虫洞，并确定了对所有亏格求和后得到的一维有效理论。我们会特别详细讲解: 对带双局域相互作用的二维可投影 HL 引力量子化即可得到该一维有效理论，并讨论二维 CDT 中的科尔曼机制。最后“小结”小节用于总结全章内容。

2d Causal Dynamical Triangulations

二维因果动力学三角剖分

Two-dimensional causal dynamical triangulations (2d CDT) [9] is a lattice model of quantum geometries based on Regge's discretization [10]. In this section, we give an overview of 2d CDT and in particular explain how to obtain the quantum Hamiltonian through the continuum limit.

二维因果动力学三角剖分 (2d CDT)[9] 是基于里奇离散化 [10] 的量子几何格子模型。本节我们概述 2d CDT，并具体说明如何通过连续极限得到量子哈密顿量。

We start with a two-dimensional globally hyperbolic manifold equipped with a global time foliation:

我们从配备了全局时间叶状结构的二维整体双曲流形开始:

$$M = \bigcup_{t \in \mathbb{R}} \sum\nolimits_t \tag{1}$$

where each leaf $\sum_t$ is a one-dimensional Cauchy "surface" (line). One approximates the manifold with a foliation in such a way that the continuous label t is discretized by integers, i.e., $t \in \mathbb{Z}$; each leaf (line) is partitioned by vertices connected by isometric edges; vertices among neighboring time steps are connected by isometric edges to form a triangulation of strip (see Fig. 1). The edges at a given time step and those connecting vertices in different time steps are, respectively, space-like and time-like edges since the squared edge lengths of the space-like edge a_s^2 and the time-like edge a_t^2 are given by

其中每个叶 $\sum_t$ 是一维柯西“曲面”(线)。我们按如下方式对叶状结构的流形做近似: 连续标签 t 被整数离散化，即 $t \in \mathbb{Z}$；每个叶 (线) 被等长边长连接的顶点分割；相邻时间步的顶点通过等长边长连接，形成带状三角剖分 (参见图 1)。给定时间步的边和连接不同时间步顶点的边分别是类空边和类时边，因为类空边 a_s^2 和类时边 a_t^2 的平方边长由下式给出

$$a_s^2 = \varepsilon^2,\ a_t^2 = -\alpha\varepsilon^2, \tag{2}$$

where α is a positive number and ε is a lattice spacing that serves as a UV cutoff.

其中 α 是正数，ε 是作为紫外截断的格子间距。

2d CDT deals with a set of restricted class of Lorentzian triangulations as discussed above. In particular, we consider that the topology of the one-dimensional universe (a graph consisting of vertices and edges at a given time) is either S^1 or $[0,1]$, and the topology will not change during (discrete) time propagation. Since the topology is fixed, the curvature term plays no role in two dimensions, and only the discrete analogue of the cosmological constant term $\Lambda_0 \int d^2x\sqrt{-g}$ is used as the lattice action of 2 d CDT:

2d CDT 处理上述受限类别的洛伦兹三角剖分集合。特别地，我们认为一维宇宙 (给定时间由顶点和边构成的图) 的拓扑为 S^1 或 $[0,1]$ ，且拓扑在 (离散) 时间演化过程中保持不变。由于拓扑固定，曲率项在二维中不起作用，仅宇宙学常数项的离散类比 $\Lambda_0 \int d^2x\sqrt{-g}$ 用作 2 d CDT 的格子作用量:

$$S_T[\lambda,\alpha] = -\frac{\lambda}{\varepsilon^2}\left(\frac{\sqrt{4\alpha+1}}{4}\varepsilon^2 n(T)\right), \tag{3}$$

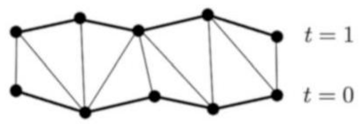
where λ/ε^2 is the bare cosmological constant with the dimensionless number $\lambda, n(T)$ the number of triangles in a triangulation T , and the term inside the parentheses denotes the total area of the triangulation. It is useful to rotate to the Euclidean signature which can be performed by changing $\alpha \to -\alpha - i0$. Accordingly, the lattice action (3) changes as follows:

其中 λ/ε^2 是裸宇宙学常数，无量纲数 $\lambda, n(T)$ 是三角剖分 T 中的三角形数量，括号内的项表示三角剖分的总面积。转到欧几里得符号是常用处理，可通过改变 $\alpha \to -\alpha - i0$ 实现。相应地，格点作用量 (3) 变换如下:

$$iS_T[\lambda,\alpha] \to iS_T[\lambda,-\alpha-i0] = -\lambda\frac{\sqrt{4\alpha-1}}{4}n(T) \equiv -\lambda n(T), \tag{4}$$

Fig. 1 A triangulation of a strip: Thick and thin lines are space-like and time-like edges, respectively

图 1 带状三角剖分: 粗线和细线分别为类空边和类时边



where α is chosen to be greater than $1/4$; otherwise, the triangle inequalities will not be satisfied after the rotation. In any case, we have absorbed the parameter α by the redefinition of the dimensionless cosmological constant λ .

其中 α 被取为大于 $1/4$ ；否则旋转后无法满足三角不等式。无论如何，我们已经通过对无量纲宇宙学常数 λ 重新定义吸收了参数 α 。

The amplitude of the one-dimensional universe that starts with ℓ_1 edges and ends up with ℓ_2 edges after the discrete time step t is given by the sum over all allowed triangulations:

从 ℓ_1 条边出发，经过离散时间步 t 后最终得到 ℓ_2 条边的一维宇宙的振幅由所有允许三角剖分的求和给出:

$$G_\lambda^{(a)}(\ell_1, \ell_2; t) = \sum_{T \in \mathcal{T}^{(a)}(\ell_1, \ell_2, t)} e^{-\lambda n(T)} = \sum_n e^{-\lambda n} \mathcal{N}^{(a)}(\ell_1, \ell_2, n), \tag{5}$$

where $\mathcal{T}^{(a)}$ is a set of triangulations whose topology is $[0,1] \times [0,1]$ for $a = 0$ and $S^1 \times [0,1]$ for $a = 1$, and

其中 $\mathcal{T}^{(a)}$ 是满足拓扑为 $[0,1] \times [0,1]$ (对应 $a = 0$)、拓扑为 $S^1 \times [0,1]$ (对应 $a = 1$) 的三角剖分集合，且

$$\mathcal{N}^{(a)}(\ell_1, \ell_2, n) = \#\left\{T \in \mathcal{T}^{(a)} \mid n(T) = n\right\}. \tag{6}$$

When defining the amplitude (5), we do not allow the one-dimensional universe to vanish during the discrete time propagation. For later convenience, we introduce a marked amplitude:

在定义振幅 (5) 时，我们不允许一维宇宙在离散时间演化过程中消失。为后续方便，我们引入标记振幅:

$$G_\lambda^{(-1)}(\ell_1, \ell_2; t) = \ell_1 G_\lambda^{(1)}(\ell_1, \ell_2; t), \tag{7}$$

where one of the edges in the initial one-dimensional universe is marked. This is because there exist ℓ_1 possible ways of marking the edges. The three kinds of amplitude should satisfy the composition law:

其中初始一维宇宙中有一条边被标记。这是因为共有 ℓ_1 种可能的标记方式。三种振幅满足组合律:

$$G_\lambda^{(1)}(\ell_1, \ell_2; t_1 + t_2) = \sum_{\ell=1}^{\infty} G_\lambda^{(1)}(\ell_1, \ell; t_1)\, \ell\, G_\lambda^{(1)}(\ell, \ell_2; t_2), \tag{8}$$

$$G_\lambda^{(0)}(\ell_1, \ell_2; t_1 + t_2) = \sum_{\ell=1}^{\infty} G_\lambda^{(0)}(\ell_1, \ell; t_1)\, G_\lambda^{(0)}(\ell, \ell_2; t_2), \tag{9}$$

$$G_\lambda^{(-1)}(\ell_1, \ell_2; t_1 + t_2) = \sum_{\ell=1}^{\infty} G_\lambda^{(-1)}(\ell_1, \ell; t_1)\, G_\lambda^{(-1)}(\ell, \ell_2; t_2), \tag{10}$$

where for $a = 1$, one needs to multiply the amputated amplitude by ℓ since there exist ℓ possible ways of gluing to recover the whole amplitude.

其中对于 $a = 1$，需要将截肢振幅乘以 ℓ，因为共有 ℓ 种粘合方式还原完整振幅。

It is convenient to introduce the generating function of the number of triangulations. Using the notation

引入三角剖分数的生成函数会更方便。使用记号

$$g = e^{-\lambda} \tag{11}$$

we define the generating function:

我们定义生成函数:

$$\begin{aligned}\widetilde{G}^{(a)}(g,x,y;t) &= \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} x^{\ell_1} y^{\ell_2} G_\lambda^{(a)}(\ell_1,\ell_2;t) \\ &= \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_n x^{\ell_1} y^{\ell_2} g^n \mathcal{N}^{(a)}(\ell_1,\ell_2,n),\end{aligned} \tag{12}$$

where in the context of quantum gravity, x and y are related to the boundary cosmological constants, λ_1 and λ_2 , that control the size of the boundaries:

在量子引力的语境下，x 与 y 和控制边界尺寸的边界宇宙学常数 λ_1 与 λ_2 相关:

$$x = e^{-\lambda_1},\ y = e^{-\lambda_2}. \tag{13}$$

One can reconstruct the amplitude from the generating function through the following relation:

我们可以通过以下关系从生成函数重构振幅:

$$G_\lambda^{(a)}(\ell_1,\ell_2;t) = \oint_{C_1} \frac{dx}{2\pi i x^{\ell_1+1}} \oint_{C_2} \frac{dy}{2\pi i y^{\ell_2+1}} \widetilde{G}^{(a)}(g,x,y;t), \tag{14}$$

where the contour $\mathcal{C}_1$ ($\mathcal{C}_2$) is chosen to enclose $x = 0$ ($y = 0$) and to ensure the convergence of $\widetilde{G}^{(a)}(g,x,y;t)$. One can derive the relation (14) using the identity:

其中选取围道 $\mathcal{C}_1$ ($\mathcal{C}_2$) 包围 $x = 0$ ($y = 0$)，并保证 $\widetilde{G}^{(a)}(g,x,y;t)$ 收敛。我们可以利用如下恒等式推导出关系式 (14):

$$\oint_C \frac{dz}{2\pi i z^{n+1}} = \delta_{n,0},\ (n \in \mathbb{Z}), \tag{15}$$

where the contour C encloses $z = 0$. We provide the composition law for the generating function when $a = 0, -1$ in preparation for later calculations:

其中围道 C 包围 $z = 0$ 。为后续计算做准备，我们给出当 $a = 0, -1$ 时生成函数的合成定律:

$$\widetilde{G}^{(a)}(g,x,y;t_1+t_2) = \oint_C \frac{dz}{2\pi i z} \widetilde{G}^{(a)}\left(g,x,z^{-1};t_1\right) \widetilde{G}^{(a)}(g,z,y;t_2), (a = 0,-1), \tag{16}$$

where the contour encloses $z = 0$ and for fixed g, x , and y lies inside the radius of convergence for $\widetilde{G}^{(a)}\left(g,x,z^{-1};t_1\right)$ as the series in $1/z$ and for $\widetilde{G}^{(a)}(g,z,y;t_2)$ as the series in z , which is possible as we will see.

其中围道包围 $z = 0$ ，且对于固定的 g, x ，y 处于 $\widetilde{G}^{(a)}\left(g,x,z^{-1};t_1\right)$ 关于 $1/z$ 的幂级数以及 $\widetilde{G}^{(a)}(g,z,y;t_2)$ 关于 z 的幂级数的收敛半径内，我们后续会看到这是成立的。

In what follows, we will discuss the one-step amplitude $G_{\lambda}^{(a)}(\ell_1, \ell_2; 1)$. This is because it becomes an important object when computing the whole amplitude.

在下文中，我们将讨论单步振幅 $G_{\lambda}^{(a)}(\ell_1, \ell_2; 1)$ ，这是因为它在计算总振幅时是一个重要对象。

Counting Triangulations

三角剖分计数

In this section, we focus on the one-step amplitude $G_{\lambda}^{(a)}(\ell_1, \ell_2; 1)$, which is the sum over triangulations of a strip as shown in Fig. 1:

本节我们聚焦于单步振幅 $G_{\lambda}^{(a)}(\ell_1, \ell_2; 1)$ ，它是对图 1 所示长条带的所有三角剖分求和:

$$G_{\lambda}^{(a)}(\ell_1, \ell_2; 1) = e^{-\lambda(\ell_1+\ell_2)}\mathcal{N}^{(a)}(\ell_1, \ell_2, n = \ell_1 + \ell_2), \tag{17}$$

and count the number of triangulations $\mathcal{N}^{(a)}(\ell_1, \ell_2, n = \ell_1 + \ell_2)$. Based on simple combinatorics, one can calculate the case of $a = 1$, which has the $S^1 \times [0, 1]$ topology:

接下来我们计数三角剖分的数量 $\mathcal{N}^{(a)}(\ell_1, \ell_2, n = \ell_1 + \ell_2)$ 。通过简单组合学，我们可以计算具有 $S^1 \times [0, 1]$ 拓扑的 $a = 1$ 情形:

$$\mathcal{N}^{(1)}(\ell_1, \ell_2, n = \ell_1 + \ell_2) = \frac{1}{\ell_1 + \ell_2}\binom{\ell_1 + \ell_2}{\ell_1} = \frac{(\ell_1 + \ell_2 - 1)!}{\ell_1!\ell_2!}. \tag{18}$$

Because of the property (7), one can easily compute the case of $a = -1$ that the topology is $S^1 \times [0, 1]$, and one of the edges in the initial one-dimensional universe is marked:

由性质 (7)，我们可以很容易计算 $a = -1$ 、拓扑为 $S^1 \times [0, 1]$ 且初始一维宇宙中有一条边被标记的情形:

$$\mathcal{N}^{(-1)}(\ell_1, \ell_2, n = \ell_1 + \ell_2) = \ell_1\mathcal{N}^{(1)}(\ell_1, \ell_2, n = \ell_1 + \ell_2) = \frac{\ell_1(\ell_1 + \ell_2 - 1)!}{\ell_1!\ell_2!}.$$

(19)

Concerning the case of $a = 0$ whose topology is $[0, 1] \times [0, 1]$, there exist several possibilities depending on the restriction on the leftmost and rightmost triangles. If the rightmost triangle is the upward triangle (downward triangle) and the leftmost triangles is the downward triangle (upward triangle), then the counting of triangulations yields

对于拓扑为 $[0, 1] \times [0, 1]$ 的 $a = 0$ 情形，存在多种可能性，取决于对最左侧和最右侧三角形的限制。若最右侧三角形为向上三角形 (向下三角形)，且最左侧三角形为向下三角形 (向上三角形)，三角剖分计数结果为

$$\mathcal{N}^{(0)}\left(\ell_1,\ell_2,n=\ell_1+\ell_2\right)=\binom{\ell_1+\ell_2-2}{\ell_1-1}=\frac{(\ell_1+\ell_2-2)!}{(\ell_1-1)!\,(\ell_2-1)!}. \tag{20}$$

In the following, we will use Eq. (20) in the case of $a = 0$ for computational simplicity (If we do not impose any restriction on the leftmost and rightmost triangles, the number of triangulations becomes $\mathcal{N}^{(0)}\left(\ell_1,\ell_2,n=\ell_1+\ell_2\right) = \left(\ell_1+\ell_2\right)!/\left(\ell_1!\ell_2!\right)$)

下文为简化计算，我们将在 $a = 0$ 的情形下使用式 (20)(若不对最左侧和最右侧三角形施加任何限制，三角剖分的数量为 $\mathcal{N}^{(0)}\left(\ell_1,\ell_2,n=\ell_1+\ell_2\right) = \left(\ell_1+\ell_2\right)!/\left(\ell_1!\ell_2!\right)$)

The one-step generating functions can be derived inserting Eqs. (18), (19), and (20) into Eq. (12):

将式 (18)、(19) 和 (20) 代入式 (12)，即可推导出一步生成函数:

$$\begin{aligned}\widetilde{G}^{(1)}\left(g,x,y;1\right) &= \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} x^{\ell_1}y^{\ell_2}g^{\ell_1+\ell_2}\mathcal{N}^{(1)}\left(\ell_1,\ell_2,n=\ell_1+\ell_2\right)\\ &= -\ln\left(\frac{1-gx-gy}{(1-gx)(1-gy)}\right);\end{aligned} \tag{21}$$

$$\begin{aligned}\widetilde{G}^{(-1)}\left(g,x,y;1\right) &= \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} x^{\ell_1}y^{\ell_2}g^{\ell_1+\ell_2}\mathcal{N}^{(-1)}\left(\ell_1,\ell_2,n=\ell_1+\ell_2\right)\\ &= \frac{g^2xy}{(1-gx)(1-gx-gy)};\end{aligned} \tag{22}$$

$$\begin{aligned}\widetilde{G}^{(0)}\left(g,x,y;1\right) &= \sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty} x^{\ell_1}y^{\ell_2}g^{\ell_1+\ell_2}\mathcal{N}^{(0)}\left(\ell_1,\ell_2,n=\ell_1+\ell_2\right)\\ &= \frac{g^2xy}{1-gx-gy}.\end{aligned} \tag{23}$$

In fact, one can also obtain Eq. (22) through $\widetilde{G}^{(-1)}\left(g,x,y;1\right) = x\frac{\partial}{\partial x}\widetilde{G}^{(1)}\ \left(g,x,y;1\right)$.

事实上，我们也可以通过 $\widetilde{G}^{(-1)}\left(g,x,y;1\right) = x\frac{\partial}{\partial x}\widetilde{G}^{(1)}\ \left(g,x,y;1\right)$ 得到式 (22)。

Alternatively, it is possible to compute the one-step generating functions directly by simple combinatorics:

另外，我们也可以通过基础组合数学直接计算一步生成函数:

$$\widetilde{G}^{(1)}\left(g,x,y;1\right) = \sum_{s=1}^{\infty}\frac{1}{s}\left(\sum_{k=1}^{\infty}(gx)^k\sum_{l=1}^{\infty}(gy)^l\right)^s = -\ln\left(\frac{1-gx-gy}{(1-gx)(1-gy)}\right);$$

(24)

$$\widetilde{G}^{(-1)}\left(g,x,y;1\right) = \sum_{k=0}^{\infty}\left(gx\sum_{l=0}^{\infty}(gy)^l\right)^k - \sum_{k=0}^{\infty}(gx)^k = \frac{g^2xy}{(1-gx)(1-gx-gy)};$$

(25)

$$\widetilde{G}^{(0)}(g,x,y;1) = \sum_{s=1}^{\infty}\frac{1}{s}\left(\sum_{k=1}^{\infty}(gx)^k\sum_{l=1}^{\infty}(gy)^l\right)^s = \frac{g^2xy}{1-gx-gy}. \tag{26}$$

Continuum Limit

连续极限

All is now set for computing the amplitude in the continuum limit. In this section, however, instead of directly computing the amplitude in the continuum limit, we will derive the differential equation that the continuum amplitude satisfies.

现在已经为计算连续极限下的振幅做好了所有准备。但在本节中，我们不会直接计算连续极限下的振幅，而是推导连续振幅满足的微分方程。

Before going into details any further, let us explain some basic facts of the continuum limit. In order to remove the cutoff ε through the continuum limit, one has to tune the bare coupling constants (g,x,y) to their critical values (g_c,x_c,y_c) . At the critical values, the generating function hits the radii of convergence and therefore becomes non-analytic. Approaching such a critical point, infinitely many triangles and boundary edges become important in the summation of the generating function, i.e., essentially, the average number of triangles and boundary edges become infinity at the critical point. Having this in mind, one may intuitively understand that the continuous surface would be obtained if $(g,x,y)\to(g_c,x_c,y_c)$ and $\varepsilon\to 0$ in a correlated manner.

在进一步展开细节之前，我们先说明连续极限的一些基本性质。为了通过连续极限移除截断 ε，必须将裸耦合常数 (g,x,y) 调整到其临界值 (g_c,x_c,y_c)。在临界值处，生成函数达到收敛半径，因此变得非解析。当趋近于该临界点时，无穷多三角形和边界边在生成函数求和中占据主导，也就是说临界点处三角形与边界边的平均数量本质上会变为无穷大。记住这一点，就可以直观理解: 当 $(g,x,y)\to(g_c,x_c,y_c)$ 和 $\varepsilon\to 0$ 按关联方式变化时，我们就能得到连续曲面。

Introducing $\lambda_c = -\ln[g_c]$, $\lambda_{1c} = -\ln[x_c]$ and $\lambda_{2c} = -\ln[y_c]$, one can transmute the dimension of the lattice spacing ε into the dimension of the renormalized bulk and boundary cosmological constants through the continuum limit:

引入 $\lambda_c = -\ln[g_c]$, $\lambda_{1c} = -\ln[x_c]$ 和 $\lambda_{2c} = -\ln[y_c]$ 后，我们可以通过连续极限将格点间距 ε 的量纲转化为重整化的体宇宙学常数和边界宇宙学常数的量纲:

$$\Lambda = \lim_{\substack{\lambda\to\lambda_c\\ \varepsilon\to 0}}\frac{\lambda-\lambda_c}{\varepsilon^2},\ X = \lim_{\substack{\lambda_1\to\lambda_{1c}\\ \varepsilon\to 0}}\frac{\lambda_1-\lambda_{1c}}{\varepsilon},\ Y = \lim_{\substack{\lambda_2\to\lambda_{2c}\\ \varepsilon\to 0}}\frac{\lambda_2-\lambda_{2c}}{\varepsilon}, \tag{27}$$

where Λ is the renormalized bulk cosmological constant and X and Y are the renormalized boundary cosmological constants. Therefore, the divergent bare cosmological constants get additive renormalizations so as to obtain the finite renormalized cosmological constants that set the scale at IR.

其中 Λ 是重整化体宇宙学常数，X 和 Y 是重整化边界宇宙学常数。因此，发散的裸宇宙学常数经过加法重整化后，得到了有限的重整化宇宙学常数，用于设定红外能标。

In the following, we discuss the continuum limit in detail with respect to each topology of spacetime.

下文我们将针对每种时空拓扑，详细讨论连续极限。

$S^1 \times [0,1]$ Topology

$S^1 \times [0,1]$ 拓扑

We consider the case of $a = -1$, i.e., $S^1 \times [0,1]$ topology with a marked boundary. Using the composition law (16) and the one-step generating function (25), one obtains

我们考虑 $a = -1$ 的情形，即带有标记边界的 $S^1 \times [0,1]$ 拓扑。利用合成律 (16) 和一步生成函数 (25)，可得

$$
\begin{aligned}
\widetilde{G}^{(-1)}(g,x,y;t+1) &= \oint_{\mathcal{C}} \frac{dz}{2\pi i z} \widetilde{G}^{(-1)}\left(g,x,z^{-1};1\right) \widetilde{G}^{(-1)}(g,z,y;t) \\
&= \oint_{C} \frac{dz}{2\pi i} \frac{g^2 x}{(1-gx)^2 (z - g/(1-gx))} \frac{\widetilde{G}^{(-1)}(g,z,y;t)}{z} \\
&= \frac{gx}{1-gx} \widetilde{G}^{(-1)}\left(g, \frac{g}{1-gx}, y; t\right).
\end{aligned} \tag{28}
$$

In the last equality, we have picked up a pole at $z = g/(1-gx)$, and there exists no pole at $z = 0$ since $\frac{\widetilde{G}^{(-1)}(g,z,y;t)}{z}$ is regular. Through iterative use of Eq. (28), one can analytically compute the generating function and extract the information of the critical point [9]. However, we do not compute the generating function directly to obtain the critical point. Instead, we follow the procedure shown in [8]: One assumes the existence of the critical point and determines the value of the critical coupling constants from the consistency.

在最后一个等式中，我们提取了 $z = g/(1-gx)$ 处的极点，由于 $\frac{\widetilde{G}^{(-1)}(g,z,y;t)}{z}$ 是正则的，$z = 0$ 处不存在极点。通过反复使用式 (28)，我们可以解析计算生成函数并提取临界点的信息 [9]。但我们并不直接计算生成函数来得到临界点，而是遵循文献 [8] 中的方法: 先假设临界点存在，再通过自洽性确定临界耦合常数的取值。

We assume the critical point characterized by the critical coupling constants (g_c, x_c, y_c) and use the following parametrization:

我们假设临界点由临界耦合常数 (g_c, x_c, y_c) 刻画，并采用如下参数化:

$$
g = g_c e^{-\varepsilon^2 \Lambda}, \; x = x_c e^{-\varepsilon X}, \; y = y_c e^{-\varepsilon Y}. \tag{29}
$$

Assuming the scalings

假设标度变换为

$$T = \varepsilon t,\ L_1 = \varepsilon \ell_1,\ L_2 = \varepsilon \ell_2, \tag{30}$$

we introduce the renormalized amplitude and the renormalized generating function at the critical point by the multiplicative renormalizations:

我们通过乘性重整化，引入临界点处的重整化振幅和重整化生成函数:

$$G_\Lambda^{(-1)}(L_1, L_2; T) = \lim_{\varepsilon \to 0} C_\varepsilon G_\lambda^{(-1)}(\ell_1, \ell_2; t), \tag{31}$$

$$\widetilde{G}_\Lambda^{(-1)}(X, Y; T) = \lim_{\varepsilon \to 0} \widetilde{C}_\varepsilon \widetilde{G}^{(-1)}(g, x, y; t), \tag{32}$$

where C_ε and $\widetilde{C}_\varepsilon$ are real functions of ε that will be fixed below. The function C_ε can be determined in such a way that the composition law (10) holds in the continuum limit as

其中 C_ε 和 $\widetilde{C}_\varepsilon$ 是关于 ε 的实函数，将在下文确定。可以通过条件确定函数 C_ε，使得连续极限下合成律 (10) 成立，即

$$G_\Lambda^{(-1)}(L_1, L_2; T_1 + T_2) = \int_0^\infty dL G_\Lambda^{(-1)}(L_1, L; T_1)\, G_\Lambda^{(-1)}(L, L_2; T_2), \tag{33}$$

which is possible if $C_\varepsilon = \varepsilon^{-1}$. The function $\widetilde{C}_\varepsilon$ can be determined in such way that Eq. (12) makes sense in the continuum limit, i.e.,

当 $C_\varepsilon = \varepsilon^{-1}$ 时该条件成立。可以通过条件确定函数 $\widetilde{C}_\varepsilon$， 使得式 (12) 在连续极限下有意义，即

$$\widetilde{G}_\Lambda^{(-1)}(X, Y; T) = \int_0^\infty dL_1 \int_0^\infty dL_2 e^{-XL_1} e^{-YL_2} G_\Lambda^{(-1)}(L_1, L_2; T), \tag{34}$$

which is possible if $\widetilde{C}_\varepsilon = \varepsilon / (x_c y_c)$.

当 $\widetilde{C}_\varepsilon = \varepsilon / (x_c y_c)$ 时该条件成立。

Using the scaling behavior (32), Eq. (28) can yield the sensible continuum limit if the critical coupling constants satisfy

利用标度行为 (32)，若临界耦合常数满足下式，则式 (28) 可得到合理的连续极限:

$$\frac{g_c x_c}{1 - g_c x_c} = 1,\ \frac{g_c}{1 - g_c x_c} = x_c \Rightarrow g_c = \frac{1}{2},\ x_c = 1. \tag{35}$$

Now we wish to take the continuum limit of Eq. (28). For notational convenience, we redefine the renormalized coupling constants as follows:

现在我们对式 (28) 取连续极限。为简化记号，我们重新定义重整化耦合常数如下:

$$g = \frac{1}{2}e^{-\varepsilon^2\Lambda} \equiv \frac{1}{2}\left(1 - \frac{1}{2}\varepsilon^2\Lambda\right), \; x = e^{-\varepsilon X} \equiv 1 - \varepsilon X, \; y = e^{-\varepsilon Y} \equiv 1 - \varepsilon Y.$$

(36)

Plugging Eqs. (30) and (36) into Eq. (28), one obtains the differential equation:

将式 (30) 和 (36) 代入式 (28)，可得微分方程:

$$\frac{\partial}{\partial T}\widetilde{G}_{\Lambda}^{(-1)}(X, Y; T) = -\frac{\partial}{\partial X}\left[\left(X^2 - \Lambda\right)\widetilde{G}_{\Lambda}^{(-1)}(X, Y; T)\right]. \tag{37}$$

Doing a little math, one can also derive the continuum description of Eq. (14):

经过简单推导，我们还可以得到式 (14) 的连续极限描述:

$$G_{\Lambda}^{(-1)}(L_1, L_2; T) = \int_{c-i\infty}^{c+i\infty}\frac{dX}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{dY}{2\pi i}e^{L_1 X}e^{L_2 Y}\widetilde{G}_{\Lambda}^{(-1)}(X, Y; T),$$

(38)

where c is a suitable real number. Using the inverse Laplace transform (38) and Eq. (37), one obtains the differential equation that the continuum amplitude satisfies:

其中 c 是一个合适的实数。利用逆拉普拉斯变换 (38) 和式 (37)，我们可以得到连续振幅满足的微分方程:

$$\frac{\partial}{\partial T}G_{\Lambda}^{(-1)}(L_1, L_2; T) = -\widehat{H}^{(-1)}(L_1)\, G_{\Lambda}^{(-1)}(L_1, L_2; T), \tag{39}$$

where

其中

$$\widehat{H}^{(-1)}(L) = -L\frac{\partial^2}{\partial L^2} + \Lambda L. \tag{40}$$

As a result, one can interpret the continuum limit of 2d CDT as a quantum system of the one-dimensional universe with length L that propagates in time T following the quantum Hamiltonian (40). The quantum Hamiltonian (40) is Hermitian with respect to the inner product:

因此，我们可以将二维 CDT 的连续极限解释为: 一维宇宙是一个量子系统，长度为 L，沿着量子哈密顿量 (40) 在时间 T 上演化。量子哈密顿量 (40) 关于如下内积是厄米的:

$$\int_0^\infty \frac{dL}{L}\phi^*(L)\left(\widehat{H}^{(-1)}\psi\right)(L) = \int_0^\infty \frac{dL}{L}\left(\widehat{H}^{(-1)}\phi\right)^*(L)\,\psi(L). \tag{41}$$

The differential equation for the un-marked amplitude can be easily read off inserting the continuum limit of Eq. (7)

将式 (7) 的连续极限代入，即可直接得到未标记振幅的微分方程

$$G_\Lambda^{(-1)}(L_1, L_2; T) = L_1 G_\Lambda^{(1)}(L_1, L_2; T), \tag{42}$$

into Eq. (39):

到式 (39) 中:

$$\frac{\partial}{\partial T} G_\Lambda^{(1)}(L_1, L_2; T) = -\widehat{H}^{(1)}(L_1) G_\Lambda^{(1)}(L_1, L_2; T), \tag{43}$$

where

其中

$$\widehat{H}^{(1)}(L) = -\frac{\partial^2}{\partial L^2} L + \Lambda L. \tag{44}$$

The quantum Hamiltonian (44) is Hermitian with respect to the inner product:

量子哈密顿量 (44) 关于以下内积是厄米的:

$$\int_0^\infty L dL \phi^*(L) \left(\widehat{H}^{(1)}\psi\right)(L) = \int_0^\infty L dL \left(\widehat{H}^{(1)}\phi\right)^*(L) \psi(L). \tag{45}$$

[0, 1] x [0, 1] Topology

[0, 1] × [0, 1] 拓扑

Let us consider the case of $a = 0$, i.e., $[0,1] \times [0,1]$ topology. We basically follow the procedure shown in section " $S^1 \times [0,1]$ Topology." Using the composition law (16) and the one-step generating function (26), one obtains

我们来考虑 $a = 0$ 的情况，即 $[0,1] \times [0,1]$ 拓扑。我们基本遵循"$S^1 \times [0,1]$ 拓扑"小节中给出的步骤。利用合成定律 (16) 和一步生成函数 (26)，可得

$$\begin{aligned}
\widetilde{G}^{(0)}(g, x, y; t+1) &= \oint_C \frac{dz}{2\pi i z} \widetilde{G}^{(0)}(g, x, z^{-1}; 1) \widetilde{G}^{(0)}(g, z, y; t) \\
&= \oint_C \frac{dz}{2\pi i} \frac{g^2 x}{(1-gx)(z - g/(1-gx))} \frac{\widetilde{G}^{(0)}(g, z, y; t)}{z} \\
&= gx \widetilde{G}^{(0)}\left(g, \frac{g}{1-gx}, y; t\right).
\end{aligned} \tag{46}$$

From Eq. (46), one may obtain a sensible continuum limit if the critical coupling constants are the same as before, i.e., $(g_c, x_c, y_c) = (1/2, 1, 1)$, and if the multiplicative renormalization is treated carefully:

根据式 (46)，若临界耦合常数与之前相同即 $(g_c, x_c, y_c) = (1/2, 1, 1)$ ，且乘性重整化处理得当，就可以得到合理的连续极限:

$$\widetilde{G}_{\Lambda}^{(0)}(X,Y;T)=\lim_{\varepsilon\to 0}\frac{\varepsilon}{2^{t}}\widetilde{G}^{(0)}(g,x,y;t)\,. \tag{47}$$

In fact, this assumption yields the correct continuum limit. Plugging Eqs. (30) and (36) into Eq. (46) and using Eq. (47), one obtains the differential equation:

事实上，该假设可以得到正确的连续极限。将式 (30) 和式 (36) 代入式 (46)，再利用式 (47)，可得微分方程:

$$\frac{\partial}{\partial T}\widetilde{G}_{\Lambda}^{(0)}(X,Y;T)=-\left(X+\left(X^{2}-\Lambda\right)\frac{\partial}{\partial X}\right)\widetilde{G}_{\Lambda}^{(0)}(X,Y;T)\,. \tag{48}$$

Defining the continuum amplitude in such a way that the inverse Laplace transform (38) holds in the case of $a=0$ as well, i.e.,

我们这样定义连续振幅，使得逆拉普拉斯变换 (38) 在 $a=0$ 的情况下同样成立，即

$$G_{\Lambda}^{(0)}(L_1,L_2;T)=\int_{c-i\infty}^{c+i\infty}\frac{dX}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{dY}{2\pi i}e^{L_1X}e^{L_2Y}\widetilde{G}_{\Lambda}^{(0)}(X,Y;T)\,, \tag{49}$$

and using Eq. (49), the differential equation (48) becomes

再结合式 (49)，微分方程 (48) 变为

$$\frac{\partial}{\partial T}G_{\Lambda}^{(0)}(L_1,L_2;T)=-\widehat{H}^{(0)}(L_1)\,G_{\Lambda}^{(0)}(L_1,L_2;T)\,, \tag{50}$$

where $\widehat{H}^{(0)}$ is the quantum Hamiltonian obtained in Refs. [31,32]:

其中 $\widehat{H}^{(0)}$ 是参考文献 [31,32] 中得到的量子哈密顿量:

$$\widehat{H}^{(0)}(L)=-\frac{\partial}{\partial L}L\frac{\partial}{\partial L}+\Lambda L. \tag{51}$$

The quantum Hamiltonian (51) is Hermitian with respect to the inner product:

量子哈密顿量 (51) 关于如下内积是厄米的:

$$\int_0^{\infty}dL\phi^{*}(L)\left(\widehat{H}^{(0)}\psi\right)(L)=\int_0^{\infty}dL\left(\widehat{H}^{(0)}\phi\right)^{*}(L)\,\psi(L)\,. \tag{52}$$

Short Summary of 2d CDT

二维因果动态三角剖分 (2d CDT) 概要

As discussed in section ”Continuum Limit,” the continuum limit of 2d CDT is described by the quantum mechanics of a one-dimensional universe with length L that propagates in time T based on the Hamiltonian $\widehat{H}^{(a)}$:

正如“连续极限”一节所述，二维因果动态三角剖分的连续极限可由一个长度为 L 、基于哈密顿量 $\widehat{H}^{(a)}$ 在时间 T 中演化的一维宇宙量子力学描述:

$$\widehat{H}^{(-1)}=-L\frac{\partial^2}{\partial L^2}+\Lambda L,\ \widehat{H}^{(1)}=-\frac{\partial^2}{\partial L^2}L+\Lambda L,\ \widehat{H}^{(0)}=-\frac{\partial}{\partial L}L\frac{\partial}{\partial L}+\Lambda L,$$

(53)

where the label a classifies the topology of the one-dimensional universe: S^1 and $[0,1]$ for $a=1$ and $a=0$, respectively. When $a=-1$, the closed one-dimensional universe is marked. Let us define the eigenstates of L as $|L\rangle_a$ that satisfy the completeness relation:

其中标记 a 对一维宇宙的拓扑进行分类: S^1 和 $[0,1]$ 分别对应 $a=1$ 和 $a=0$ 。当 $a=-1$ 时，对应闭合一维宇宙。我们将 L 的本征态定义为满足完备性关系的 $|L\rangle_a$:

$$\hat{1}=\int_0^\infty L^a dL|L\rangle_{aa}\langle L| \ \Leftrightarrow\ {}_a\langle L'|L\rangle_a=\frac{1}{L^a}\delta\left(L-L'\right). \tag{54}$$

Note that $|L\rangle_{-1}=L|\,L\rangle_1$. One can then express the amplitudes as matrix elements:

注意 $|L\rangle_{-1}=L|\,L\rangle_1$ 。随后我们可以将振幅表示为矩阵元:

$$G_\Lambda^{(1)}\left(L_1,L_2;T\right)={}_1\left\langle L_2\left|e^{-T\widehat{H}^{(1)}}\right|L_1\right\rangle, \tag{55}$$

$$G_\Lambda^{(-1)}\left(L_1,L_2;T\right)={}_1\left\langle L_2\left|e^{-T\widehat{H}^{(-1)}}\right|L_1\right\rangle_{-1}, \tag{56}$$

$$G_\Lambda^{(0)}\left(L_1,L_2;T\right)={}_0\left\langle L_2\left|e^{-T\widehat{H}^{(0)}}\right|L_1\right\rangle_0, \tag{57}$$

Using Eq. (54), one can show that the composition laws hold: For $a=-1,0$,

利用式 (54) 可证明组合律成立: 对于 $a=-1,0$,

$$G_\Lambda^{(a)}\left(L_1,L_2;T_1+T_2\right)=\int_0^\infty dL G_\Lambda^{(a)}\left(L_1,L;T_1\right)G_\Lambda^{(a)}\left(L,L_2;T_2\right), \tag{58}$$

and for $a=1$,

而对于 $a=1$,

$$G_\Lambda^{(1)}\left(L_1,L_2;T_1+T_2\right)=\int_0^\infty dL_\Lambda^{(1)}\left(L_1,L;T_1\right)LG_\Lambda^{(1)}\left(L,L_2;T_2\right). \tag{59}$$

2d Projectable Hořava-Lifshitz Quantum Gravity

二维可投影霍拉瓦-利夫希茨量子引力

We wish to introduce the classical field theory that reproduces the continuum limit of 2d CDT once it is quantized. The field theory is a certain version of the two-dimensional Hořava-Lifshitz gravity (2d HL gravity).

我们旨在介绍这个经量子化后可重现二维 CDT 连续极限的经典场论。该场论是二维霍拉瓦-利夫希茨引力 (二维 HL 引力) 的一个特定版本。

The starting point is the same class of manifold with a foliation (1) where $\sum_t$ is a one-dimensional space labelled by t :

我们的出发点是同一类带叶状结构的流形 (1)，其中 $\sum_t$ 是由 t 标记的一维空间:

$$\sum_t = \{x^\mu \in \mathcal{M} \mid f(x^\mu) = t\}, \quad \text{with} \quad \mu = 0, 1. \tag{60}$$

Choosing that $f(x^\mu) = x^0$, the time direction can be decomposed into the two directions, i.e., the normal and the tangential to $\sum_t$:

选定 $f(x^\mu) = x^0$ 后，时间方向可分解为垂直于 $\sum_t$ 的法向和切向两个方向:

$$(\partial_t)^\mu = \frac{\partial x^\mu}{\partial t} = N n^\mu + N^1 E_1^\mu, \tag{61}$$

where n^μ and E_1^μ are, respectively, the unit normal vector and the tangent vector defined as

其中 n^μ 和 E_1^μ 分别是单位法向量和切向量，定义为

$$n^\mu = \left(\frac{1}{N}, -\frac{N^1}{N}\right), \; E_1^\mu = \delta_1^\mu. \tag{62}$$

Here N and N^1 are the Lapse function and the shift vector. Through the use of Eq. (61), one can parametrize the metric $g_{\mu\nu}$ on $\mathcal{M}$ as follows:

此处 N 和 N^1 分别是时移函数和位移矢量。利用式 (61)，我们可以将 $\mathcal{M}$ 上的度量 $g_{\mu\nu}$ 参数化为如下形式:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\upsilon = -N^2 dt^2 + h_{11}\left(dx + N^1 dt\right)\left(dx + N^1 dt\right), \tag{63}$$

where $t = x^0$ and $x = x^1$; h_{11} is the spatial metric on $\sum_t$ defined as $h_{11} = E_1^\mu E_1^\upsilon g_{\mu\upsilon}$

其中 $t = x^0$ 和 $x = x^1$; h_{11} 是定义在 $\sum_t$ 上的空间度量，即 $h_{11} = E_1^\mu E_1^\upsilon g_{\mu\upsilon}$

2d HL gravity is a field theory that preserves the structure of the time foliation, or in other words, it is invariant under the foliation-preserving diffeomorphisms (FPD):

二维 HL 引力是保留时间叶状结构的场论，换言之，它在保叶状结构微分同胚 (FPD) 下保持不变:

$$t \to t + \xi^0(t), \; x \to x + \xi^1(t,x). \tag{64}$$

The fields transform under FPD as follows:

场在 FPD 下的变换如下:

$$\delta_\xi h_{11} = \xi^0 \partial_0 h_{11} + \xi^1 \partial_1 h_{11} + 2 h_{11} \partial_1 \xi^1, \tag{65}$$

$$\delta_\xi N_1 = \xi^\mu \partial_\mu N_1 + N_1 \partial_\mu \xi^\mu + h_{11} \partial_0 \xi^1, \tag{66}$$

$$\delta_\xi N = \xi^\mu \partial_\mu N + N \partial_0 \xi^0. \tag{67}$$

where $N_1 = h_{11} N^1$. Here if a function is a constant on each foliation $\sum_t$, such a function is called projectable. In fact, implementing FPD, the projectable Lapse function, i.e., $N = N(t)$, stays as a function only of time. The HL gravity with the projectable Lapse function is dubbed the projectable HL gravity. Since it is 2d projectable HL gravity that reproduces the continuum limit of 2 d CDT once it is quantized, from now we focus on this special version of HL gravity.

其中 $N_1 = h_{11} N^1$ 。若一个函数在每片叶状结构 $\sum_t$ 上均为常数，则称该函数是可投影的。实际上，实施 FPD 后，可投影时移函数即 $N = N(t)$ 仍仅为时间的函数。带可投影时移函数的 HL 引力被称为可投影 HL 引力。由于正是二维可投影 HL 引力经量子化后重现了 2 d CDT 的连续极限，接下来我们将聚焦于 HL 引力的这一特殊版本。

The action of 2 d projectable HL gravity is given by

2 d 可投影 HL 引力的作用量由下式给出

$$I = \int dt \mathcal{L} = \frac{1}{\kappa} \int dt dx N(t) \sqrt{h(t,x)} \left((1-\eta) K^2(t,x) - 2\widetilde{\Lambda} \right), \tag{68}$$

where $\mathcal{L}$ is the Lagrangian; $\eta, \widetilde{\Lambda}$, and κ are a dimensionless parameter, the cosmological constant, and the (dimensionless) gravitational coupling constant, respectively; h is the determinant of the spatial metric h_{11} , i.e., $h = h_{11}; K$ is the trace of the extrinsic curvature K_{11} given by

其中 $\mathcal{L}$ 是拉格朗日量; $\eta, \widetilde{\Lambda}$ 、κ 分别是无量纲参数、宇宙学常数和 (无量纲) 引力耦合常数; h 是空间度量 h_{11} 的行列式，即 $h = h_{11}; K$ 是外曲率 K_{11} 的迹，由下式给出

$$K_{11} = \frac{1}{2N} (\partial_0 - 2\nabla_1 N_1), \quad \text{with} \quad \nabla_1 N_1 = \partial_1 N_1 - \Gamma^1_{11} N_1. \tag{69}$$

Here Γ^1_{11} is the spatial Christoffel symbol:

此处 Γ^1_{11} 是空间克里斯托费尔符号:

$$\Gamma^1_{11} = \frac{1}{2}h^{11}\partial_1 h_{11} \tag{70}$$

One can in principle add higher spatial derivative terms to the action (68). However, such terms are not necessary because the model is renormalizable in two dimensions without introducing them, and therefore we omit such terms.

原则上可以在作用量 (68) 中加入更高阶空间导数项。但这些项并非必要，因为该模型在不引入它们的情况下于二维仍然可重整，因此我们省略了这类项。

The continuum limit of 2d CDT can be precisely obtained if quantizing 2d projectable HL gravity with the following identification of parameters:

对二维可投影 HL 引力量子化后，通过如下参数匹配可以精确得到二维 CDT 的连续极限:

$$\Lambda = \frac{\widetilde{\Lambda}}{2(1-\eta)},\ \eta < 1,\ \kappa = 4(1-\eta), \tag{71}$$

where Λ is the renormalized cosmological constant of CDT defined by Eq. (36).

其中 Λ 是式 (36) 定义的 CDT 重整化宇宙学常数。

Quantization

量子化

Let us overview the quantization of 2 d projectable HL gravity shown in Ref. [15] (see also Ref. [17] for another article examining this issue).

下面我们概述文献 [15] 中给出的 2 d 可投影霍拉瓦-里夫希茨引力的量子化过程 (关于该问题的另一研究可参见文献 [17])。

We introduce the conjugate momentum of $\sqrt{h}$ as Π , which satisfy the Poisson bracket:

我们引入 $\sqrt{h}$ 的共轭动量为 Π ，它满足如下泊松括号:

$$\left\{\sqrt{h(t,x)}, \Pi(t,x')\right\} = \delta(x-x'). \tag{72}$$

Through the Legendre transformation of the Lagrangian (68), one obtains the Hamiltonian of 2d projectable HL gravity:

对拉格朗日量 (68) 做勒让德变换，即可得到二维可投影霍拉瓦-里夫希茨引力的哈密顿量:

$$H = \int dx\left(\Pi(t,x)\,\partial_t\sqrt{h(t,x)}\right) - \mathcal{L} = N(t)\,\mathcal{C}(t) + \int dx N_1(t,x)\,\mathcal{C}^1(t,x).$$

(73)

Since 2d projectable HL gravity is a singular system due to the invariance under FPD, there exist two kinds of constraint:

由于 FPD 不变性，二维可投影霍拉瓦-里夫希茨引力是一个奇异系统，存在两类约束:

$$C^1(t,x) = -\frac{\partial_1 \Pi(t,x)}{\sqrt{h(t,x)}} \approx 0, \tag{74}$$

$$\mathcal{C}(t) = \int dx \left(\frac{\kappa}{4(1-\eta)} \Pi^2(t,x) \sqrt{h(t,x)} + \frac{2}{\kappa} \widetilde{\Lambda} \sqrt{h(t,x)} \right) \approx 0, \tag{75}$$

where $C^1(t,x) \approx 0$ is the momentum constraint and $C(t) \approx 0$ is the Hamiltonian constraint, which is global because of the projectable Lapse function (The Hamiltonian and the momentum constraints come from the consistency conditions that the primary constraints, $\Pi_N \approx 0$ and $\Pi_{N_1} \approx 0$, should be preserved under the time flow where Π_N and Π_{N_1} are the conjugate momenta of N and N_1 , respectively.).

其中 $C^1(t,x) \approx 0$ 是动量约束，$C(t) \approx 0$ 是哈密顿约束；由于可投影条件对推移函数的限制，哈密顿约束是整体约束 (哈密顿约束与动量约束来自原初约束 $\Pi_N \approx 0$ 和 $\Pi_{N_1} \approx 0$ 在时间流中需保持守恒的一致性条件，其中 Π_N 和 Π_{N_1} 分别是 N 和 N_1 的共轭动量)。

The strategy is to solve the momentum constraint (74) at the level of classical theory, i.e.,

我们的方案是在经典理论层面求解动量约束 (74)，即:

$$C^1(t,x) = 0 \Rightarrow \Pi(t,x) = \Pi(t), \tag{76}$$

meaning that the conjugate momentum becomes a function only of time. Applying Eq. (76), the Hamiltonian (73) reduces to the one for the one-dimensional system:

这意味着共轭动量仅为时间的函数。代入式 (76) 后，哈密顿量 (73) 退化为一维系统的哈密顿量:

$$H = N(t) \left(\frac{\kappa}{4(1-\eta)} \Pi^2(t) L(t) + \frac{2}{\kappa} \widetilde{\Lambda} L(t) \right), \text{ with } L(t) = \int dx \sqrt{h(t,x)},$$

(77)

where $L(t)$ is the invariant length of the one-dimensional universe. Let us discuss solutions to the Hamiltonian constraint. If $(\eta - 1)\widetilde{\Lambda} > 0$, one has a solution:

其中 $L(t)$ 是一维宇宙的不变长度。下面我们讨论哈密顿约束的解。若 $(\eta - 1)\widetilde{\Lambda} > 0$ ，可得解:

$$\Pi^2 = \frac{8(\eta - 1)}{\kappa^2} \widetilde{\Lambda}, \tag{78}$$

which means that the extrinsic curvature is a constant. On the other hand, if $(\eta - 1)\widetilde{\Lambda} < 0$, the only solution is

这意味着外曲率是常数。另一方面，若 $(\eta - 1)\widetilde{\Lambda} < 0$ ，唯一解为

$$L = 0\,. \tag{79}$$

Hereafter we apply the parametrization (71): We choose $(\eta - 1)\widetilde{\Lambda} < 0$, set the unimportant dimensionless gravitational constant as $\varkappa = 4(1-\eta)$, and redefine the cosmological constant as $\Lambda = \frac{\widetilde{\Lambda}}{2(1-\eta)}$. Since $\varkappa > 0$, this means that $\eta < 1$, which selects the correct sign of the kinetic term and the positive cosmological constant $\widetilde{\Lambda} > 0$. The dynamics of the classical 1 d system with the Hamiltonian (77) can be alternatively described by the following action:

接下来我们采用参数化 (71): 我们选取 $(\eta - 1)\widetilde{\Lambda} < 0$，将不重要的无量纲引力常数设为 $\varkappa = 4(1-\eta)$，并重新定义宇宙学常数为 $\Lambda = \frac{\widetilde{\Lambda}}{2(1-\eta)}$。由于 $\varkappa > 0$，这意味着 $\eta < 1$，由此可得动能项的正确符号以及正宇宙学常数 $\widetilde{\Lambda} > 0$。哈密顿量为 (77) 的经典 1 d 系统的动力学也可以用如下作用量描述:

$$S = \int_0^1 dt \left(\frac{\dot{L}^2(t)}{4N(t)L(t)} - \Lambda N(t)L(t) \right), \tag{80}$$

where $\dot{L}(t) := \frac{d}{dt}L(t)$. We then introduce the proper time:

其中 $\dot{L}(t) := \frac{d}{dt}L(t)$。接下来我们引入固有时:

$$\tau(s) = \int_0^s dt N(t)\,,\ s \in [0,1]\,. \tag{81}$$

Since the proper time (81) is invariant under the reparametrization of time, $t \to\ t + \xi^0(t)$, if one fixes the Lapse function as $N(\tau) = 1$, the length of the one-dimensional universe $L(\tau)$ is also invariant under the time redefinition. Therefore, it makes sense to discuss the amplitude such that the one-dimensional universe with the length $L_1 := L(\tau = 0)$ propagates in the proper time τ and ends up with the universe whose length is given by $L_2 := L_2(\tau = T)$.

由于固有时 (81) 在时间重参数化下不变, 即 $t \to\ t+\xi^0(t)$, 因此若将推移函数固定为 $N(\tau) = 1$, 一维宇宙的长度 $L(\tau)$ 也在时间重定义下不变。因此, 我们可以讨论如下跃迁振幅: 长度为 $L_1 := L(\tau = 0)$ 的一维宇宙在固有时 τ 中演化，最终成为长度为 $L_2 := L_2(\tau = T)$ 的一维宇宙。

With this understanding, we consider such an amplitude based on the path integral. For convenience, we rotate $t \to it$, which is possible, thanks to the foliation, and introduce the Euclidean action:

基于这一认识，我们基于路径积分研究该跃迁振幅。为方便起见，我们对 $t \to it$ 做 Wick 转动，得益于叶状结构这是可行的，随后引入欧几里得作用量:

$$S_E = \int_0^1 dt \left(\frac{\dot{L}^2(t)}{4N(t)L(t)} + \Lambda N(t)L(t) \right). \tag{82}$$

Using the Euclidean action (82), the amplitude becomes

利用欧几里得作用量 (82)，跃迁振幅可写为

$$\mathcal{G}_{\Lambda}(L_1, L_2; T) = \int \frac{\mathcal{D}N(t)}{\mathrm{Diff}[0,1]} \int_{L(0)=L_1}^{L(1)=L_2} \mathcal{D}L(t)\, e^{-S_E[N(t),L(t)]}, \tag{83}$$

where

其中

$$T := \int_0^1 dt N(t). \tag{84}$$

We fix the Lapse function as $N(\tau) = 1$ introducing the corresponding Faddeev-Popov (FP) determinant. Since the FP determinant only gives an overall constant, we will omit it in the following. After the gauge fixing, the amplitude (83) becomes

我们通过引入相应的法捷耶夫-波波夫 (FP) 行列式将偏移函数固定为 $N(\tau) = 1$。由于 FP 行列式仅给出一个整体常数，我们在下文中将其省略。规范固定后，振幅 (83) 变为

$$\mathcal{G}_{\Lambda}(L_1, L_2; T) = \int_{L(0)=L_1}^{L(T)=L_2} \mathcal{D}L(\tau) \exp\left[-\int_0^T d\tau \left(\frac{\dot{L}^2(\tau)}{4L(\tau)} + \Lambda L(\tau)\right)\right],$$

(85)

where $\dot{L}(\tau) := \frac{d}{d\tau} L(\tau)$.

其中 $\dot{L}(\tau) := \frac{d}{d\tau} L(\tau)$。

So far, we have not specified the integral measure $\mathcal{D}L(\tau)$. We apply the three kinds of measure given by

到目前为止，我们尚未确定积分测度 $\mathcal{D}L(\tau)$。我们采用如下给出的三种测度:

$$\mathcal{D}^{(a)}L(\tau) = \prod_{\tau=0}^{\tau=T} L^a(\tau)\, dL(\tau), \ (a = 0, \pm 1). \tag{86}$$

Accordingly, we consider the three kinds of amplitude, i.e., $\mathcal{G}_{\Lambda}^{(a)}(L_1, L_2; T)$, and rewrite them introducing the quantum Hamiltonian $\hat{H}^{(a)}$:

相应地，我们考虑三种振幅，即 $\mathcal{G}_{\Lambda}^{(a)}(L_1, L_2; T)$，并通过引入量子哈密顿量 $\hat{H}^{(a)}$ 将它们改写为:

$$\mathcal{G}_{\Lambda}^{(a)}(L_1, L_2; T) = {}_a\left\langle L_2 \left| e^{-T\hat{H}^{(a)}} \right| L_1 \right\rangle_a, \tag{87}$$

where the eigenstates of L satisfy the completeness relation:

其中 L 的本征态满足完备性关系:

$$1 = \int_0^\infty L^a dL\, |L\rangle_{aa}\langle L| \Leftrightarrow {}_a\langle L' \mid L\rangle_a = \frac{1}{L^a}\delta(L - L'). \tag{88}$$

In order to read off the quantum Hamiltonian $\widehat{H}^{(a)}$, we discretize the proper time interval in steps of ε and calculate the one-step matrix element $G_{\Lambda}^{(a)}(L, L'; \varepsilon)$. The normalization can be fixed so as to satisfy the following equation:

为了推导出量子哈密顿量 $\widehat{H}^{(a)}$ ，我们将固有时间区间以 ε 为步长离散化，计算单步矩阵元 $G_{\Lambda}^{(a)}(L, L'; \varepsilon)$ 。可以将归一化固定为满足下式:

$$\lim_{\varepsilon \to 0} \int_0^{\infty} L^a dL \mathcal{G}_{\Lambda}^{(a)}(L, L'; \varepsilon) = 1, \tag{89}$$

which comes from the completeness relation (88). The result is

该式来自完备性关系 (88)。结果为

$$\mathcal{G}_{\Lambda}^{(a)}(L, L'; \varepsilon) = \frac{(LL')^{(1-a)/2}}{L'\sqrt{4\pi\varepsilon L'}} e^{-\frac{(L-L')^2}{4\varepsilon L'} - \Lambda\varepsilon L'}. \tag{90}$$

Integrating the one-step amplitude together with a function, $\psi_a(L) = {}_a\langle L \mid \psi\rangle$, for $\varepsilon \ll 1$, one can read off the quantum Hamiltonian:

对单步振幅和函数 $\psi_a(L) = {}_a\langle L \mid \psi\rangle$ 关于 $\varepsilon \ll 1$ 积分后，即可得到量子哈密顿量:

$$\psi_a(L'; \varepsilon) = {}_a\left\langle L' \left| e^{-\varepsilon \widehat{H}^{(a)}} \right| \psi \right\rangle$$

$$= \int_0^{\infty} L^a dL_a \left\langle L' \left| e^{-\varepsilon \widehat{H}^{(a)}} \right| L \right\rangle_{aa}\langle L \mid \psi\rangle$$

$$\cong \psi_a(L') - \varepsilon \widehat{H}^{(a)} \psi_a(L') + \mathcal{O}\left(\varepsilon^{3/2}\right). \tag{91}$$

Using Eq. (90), one obtains

利用式 (90) 可得

$$\widehat{H}^{(-1)}(L) = -L\frac{d^2}{dL^2} + \Lambda L,\ \widehat{H}^{(0)}(L) = -\frac{d}{dL}L\frac{d}{dL} + \Lambda L,\ \widehat{H}^{(1)}(L) = -\frac{d^2}{dL^2}L + \Lambda L,$$

(92)

The quantum Hamiltonians (92) obtained by quantizing 2 d projectable HL gravity are precisely equivalent to those obtained by the continuum limit of 2 d CDT (see Eqs. (40), (44), and (51)). The amplitudes are related as follows:

通过量子化 2 d 可投影霍拉瓦-利夫希茨引力得到的量子哈密顿量 (92)，与通过 2 d 因果动态三角剖分连续极限得到的量子哈密顿量完全等价 (见式 (40)、(44) 和 (51))。振幅满足如下关系:

$$\mathcal{G}_{\Lambda}^{(-1)}(L, L'; T) = L' G_{\Lambda}^{(-1)}(L, L'; T)$$

$$\mathcal{G}_{\Lambda}^{(a)}\left(L,L';T\right)=G_{\Lambda}^{(a)}\left(L,L';T\right),\ (a=0,1).\tag{93}$$

Thereby, we understand that the classical field theory that reproduces the continuum limit of 2 d CDT once it is quantized is indeed 2 d projectable HL gravity. The projectable Lapse function allows us to introduce the reparametrization-invariant proper time and to reduce the 2 d field theory to the 1 d system.

由此我们可知，量子化后能重现 2 d 因果动态三角剖分连续极限的经典场论确实就是 2 d 可投影霍拉瓦-利夫希茨引力。可投影偏移函数允许我们引入重新参数化不变的固有时间，将 2 d 场论约化为 1 d 系统。

Sum over All Wormholes and Baby Universes

对所有虫洞与婴儿宇宙求和

In the CDT model, the spatial topology change is not allowed to occur by definition. One can generalize the 2d CDT model in such a way that spatial topology changes do occur in keeping with the foliation structure, and the universality class is the same as that of 2d CDT. Such a model is called generalized CDT (GCDT). GCDT can be constructed as both discretized and continuum models. Here of course the continuum model can be obtained by the continuum limit of the discretized model, but one can directly construct the continuum GCDT model promoting the one-dimensional quantum-mechanical system discussed in section ”2d Causal Dynamical Triangulations” to a 2d field theory that includes the splitting and joining interactions of the one-dimensional spatial universe. Such a field theory is dubbed the string field theory for CDT, in which the string means the one-dimensional universe [22]. In this section, we introduce the string field theory for CDT and briefly explain the fact that one can take the sum over all wormholes (i.e., handles) and baby universes [25, 26]. Here the baby universe is a portion of geometry that is pinched off from the ”parent universe” and vanishes into the vacuum. We also introduce an effective one-body theory that reproduces the many-body effects coming from the splitting and joining interactions. We then discuss those effects in the context of HL gravity [28]. In the end, we show that a sort of Coleman’s mechanism works when taking into account all contributions of wormholes and baby universes non-perturbatively [30].

在 CDT 模型中,按定义不允许空间拓扑发生改变。我们可以将二维 CDT 模型推广,使得空间拓扑变化能够在保持叶状结构的前提下发生，其普适类与二维 CDT 一致。这类模型被称为广义 CDT(GCDT)。GCDT 既可以构造成离散模型，也可以构造成连续模型。当然，这里的连续模型可以通过离散模型的连续极限得到，但也可以直接构造连续 GCDT 模型: 将“二维因果动态三角剖分”一节讨论的一维量子力学系统提升为二维场论，纳入一维空间宇宙的分裂与融合相互作用。这类场论被称为 CDT 的弦场论，其中的弦指的就是一维宇宙 [22]。本节我们将介绍 CDT 弦场论，简要说明可以对所有虫洞 (即手柄) 和婴儿宇宙求和的结论 [25, 26]。此处婴儿宇宙指从“母宇宙”收缩脱离、消失到真空中的一小块几何区域。我们还会介绍一个有效的单体理论，它可以重现分裂与融合相互作用带来的多体效应。之后我们会在 HL 引力的框架下讨论这些效应 [28]。最后我们将证明，当非微扰地计入所有虫洞和婴儿宇宙的贡献时，会产生一类科尔曼机制 [30]。

We introduce an operator that creates a marked closed string (i.e., a marked closed one-dimensional universe) with length $L,\Psi^{\dagger}(L)$, and an operator that annihilates a length- L closed string without a mark, $\Psi(L)$. These operators satisfy the following commutators:

我们引入一个算符，用来产生长度为 $L, \Psi^{\dagger}(L)$ 的带标记闭弦 (即带标记的一维闭宇宙)，再引入一个湮灭长度为 L 的无标记闭弦的算符 $\Psi(L)$ 。这些算符满足如下对易关系:

$$\left[\Psi(L), \Psi^{\dagger}(L')\right] = \delta(L-L'),\ [\Psi(L), \Psi(L')] = \left[\Psi^{\dagger}(L), \Psi^{\dagger}(L')\right] = 0. \tag{94}$$

The vacuum state $|\,\mathrm{vac}\,\rangle$ is defined by the equation: $\Psi(L)|\,\mathrm{vac}\,\rangle = 0$. The CDT amplitude (57) can be expressed by sandwiching the one-body Hamiltonian:

真空态 $|\,\mathrm{vac}\,\rangle$ is defined by the equation: $\Psi(L)|\,\mathrm{vac}\,\rangle = 0$ 。CDT 振幅 (57) 可以表示为单体哈密顿量的三明治形式:

$$G_{\Lambda}^{(-1)}(L_1, L_2; T) = \left\langle \mathrm{vac}\right|\ \Psi(L_2)\, e^{-T\mathcal{H}^{(-1)}{}_{\mathrm{free}}(L_1)}\Psi^{\dagger}(L_1)\,|\,\mathrm{vac}\left.\right\rangle, \tag{95}$$

where

其中

$$\mathcal{H}^{(-1)}_{\mathrm{free}}(L) = \int_0^{\infty} \frac{dL}{L}\Psi^{\dagger}(L)\left(-L\frac{\partial^2}{\partial L^2} + \Lambda L\right)\Psi(L). \tag{96}$$

Hereafter we omit the superscript (-1) for avoiding notational complexity. Adding splitting and joining interactions into the free Hamiltonian (96), one obtains the full Hamiltonian of the string field theory for CDT:

为简化记号，我们在后文省略上标 (-1)。在自由哈密顿量 (96) 中加入分裂与融合相互作用，就得到 CDT 弦场论的完整哈密顿量:

$$\begin{aligned}\mathcal{H} = \mathcal{H}_{\mathrm{free}} - g_s \int_0^{\infty} dL_1 \int_0^{\infty} dL_2 \Psi^{\dagger}(L_1)\Psi^{\dagger}(L_2)(L_1+L_2)\Psi(L_1+L_2)\\ -\alpha g_s \int_0^{\infty} dL_1 \int_0^{\infty} dL_2 \Psi^{\dagger}(L_1+L_2) L_1 \Psi(L_1) L_2 \Psi(L_2)\\ -\int_0^{\infty} dL \delta(L)\Psi(L)\end{aligned} \tag{97}$$

where the second, third, and fourth terms, respectively, mean the splitting interaction with the string coupling constant g_s , the joining interaction with the coupling constant αg_s , and the term associated with a string vanishing into the vacuum. Here the parameter α is introduced for counting the number of handles (i.e., wormholes). One can in principle calculate the amplitude for the process such that m closed strings propagate in time and end up with n closed strings:

其中第二、第三、第四项分别是带弦耦合常数 g_s 的分裂相互作用、带耦合常数 αg_s 的融合相互作用，以及与弦消失到真空中相关的项。这里参数 α 被用来统计手柄 (即虫洞) 的数量。原则上我们可以计算如下过程的振幅: m 个闭弦随时间演化，最终变为 n 个闭弦:

$$A(L_1, \cdots, L_m; L_1', \cdots, L_n'; T)$$

$$= \left\langle \text{vac} \left| \Psi\left(L_1'\right) \cdots \Psi\left(L_n'\right) e^{-T\mathcal{H}} \Psi^{\dagger}\left(L_1\right) \cdots \Psi^{\dagger}\left(L_m\right) \right| \text{vac} \right\rangle \tag{98}$$

Effective Theory

有效理论

Let us consider the full propagator $A(L_1; L_2; T)$ that includes the sum over all genera and baby universes. We can set $\alpha = 1$ without loss of generality since the parameter α plays a supplementary role and we are interested in taking the sum over all genus contributions. Somewhat miraculously, the full propagator defined in the many-body system with the Hamiltonian (97) can be effectively described by the one-body system [25, 26] :

我们来考虑包含所有亏格与婴儿宇宙求和的完全传播子 $A(L_1; L_2; T)$ 。我们可以不失一般性地令 $\alpha = 1$ ，因为参数 α 仅起辅助作用，而我们关注的是对所有亏格贡献求和。十分神奇的是，这个定义在哈密顿量 (97) 描述的多体系统中的完全传播子，可以被单粒子系统 [25, 26] 有效描述:

$$A(L_1; L_2; T) = \left\langle L_2 \left| e^{-T\hat{H}_{\text{eff}}(L_1)} \right| L_1 \right\rangle, \tag{99}$$

where $\hat{H}$ is the effective Hamiltonian given by

其中 $\hat{H}$ 是如下给出的有效哈密顿量

$$\hat{H}_{\text{eff}}(L) = -L\frac{d^2}{dL^2} + \Lambda L - g_s L^2. \tag{100}$$

This is possible because there exists a bijection called Ambjørn-Budd bijection [24] such that one can map each geometry generated in GCDT to a branched polymer with loops at the discrete level. The last term in the Hamiltonian (100), $-g_s L^2$, expresses all the effects originated with the baby universes and the wormholes. Note that the Hamiltonian (100) is not bounded from below because of the last term, but in fact this system is known to be ”classical incomplete,” which means that the Hamiltonian has discrete energy spectra and a set of square integrable eigenfunctions (see, e.g., Ref. [33] for a pedagogical explanation about the classical incomplete systems). A similar deformation has been observed in the $c = 1$ noncritical string theory [34, 35] .

这之所以可行，是因为存在一个称为安比约恩-巴德双射的对应关系 [24]，可以将 GCDT 中生成的每个几何映射到离散层面带圈的分支聚合物。哈密顿量 (100) 的最后一项 $-g_s L^2$ ，描述了婴儿宇宙和虫洞产生的所有效应。注意哈密顿量 (100) 因最后一项而下无界，但实际上该系统是“经典不完备”的，这意味着哈密顿量具有离散能谱和一组平方可积本征函数 (关于经典不完备系统的教学性解释参见例如文献 [33])。类似的形变也在 $c = 1$ 非临界弦理论 [34, 35] 中被观测到。

The full propagator (100) can be also described in terms of the path-integral:

完全传播子 (100) 也可以用路径积分描述:

$$A\left(L_1;L_2;T\right)=\int_{L(0)=L_1}^{L(T)=L_2}\mathcal{D}L\left(\tau\right)\exp\left[-\int_0^T d\tau\left(\frac{\dot{L}^2\left(\tau\right)}{4L\left(\tau\right)}+\Lambda L\left(\tau\right)-g_s L^2\left(\tau\right)\right)\right],$$

(101)

where the integral measure is given by

其中积分测度由下式给出

$$\mathcal{D}L\left(\tau\right)=\prod_{\tau=0}^{\tau=T}L^{-1}\left(\tau\right)dL\left(\tau\right). \tag{102}$$

In order for the functional integral (101) to be well defined, one needs to choose the boundary conditions on $L\left(\tau\right)$ at infinity such that the kinetic term counteracts the unboundedness of the potential. If one generalizes the integral measure (102) as

为了让泛函积分 (101) 良定义，需要对无穷远处的 $L\left(\tau\right)$ 选取边界条件，使得动能项抵消势能的无界性。如果将积分测度 (102) 推广为

$$\mathcal{D}^{(a)}L\left(\tau\right)=\prod_{\tau=0}^{\tau=T}L^{a}\left(\tau\right)dL\left(\tau\right),\ \left(a=0,\pm1\right), \tag{103}$$

one can recover all possible orderings of the effective Hamiltonian (100) following the procedure explained in section ”Quantization.”

我们就可以按照“量子化”一节介绍的步骤，得到有效哈密顿量 (100) 所有可能的排序。

Interestingly, one can reproduce the full propagator (101) if one considers that the cosmological constant Λ in Eq. (85) is not a constant but fluctuates independently in time around Λ , following the Gaussian distributions with a standard deviation $\sigma=2\sqrt{g_s}$:

有意思的是，如果我们认为式 (85) 中的宇宙学常数 Λ 不是常数，而是围绕 Λ 随时间独立涨落，遵循标准差为 $\sigma=2\sqrt{g_s}$ 的高斯分布，就可以重现完全传播子 (101):

$$A\left(L_1;L_2;T\right)=\int\mathcal{D}v\left(\tau\right)e^{-\frac{1}{4g_s}\int_0^T d\tau v^2(\tau)}\mathcal{G}_{\Lambda+v}\left(L_1,L_2;T\right), \tag{104}$$

where

其中

$$\mathcal{G}_{\Lambda+v}\left(L_1,L_2;T\right)\ :=\ \int_{L(0)=L_1}^{L(T)=L_2}\mathcal{D}L\left(\tau\right)\exp\left[-\int_0^T d\tau\left(\frac{\dot{L}^2\left(\tau\right)}{4L\left(\tau\right)}+\left(\Lambda+v\left(\tau\right)\right)L\left(\tau\right)\right)\right].$$

(105)

Therefore, all the contributions coming from the sum over all wormholes and baby universes can be fully taken in if the cosmological ”constant” in (the continuum limit of) 2d CDT or projectable HL quantum gravity where no wormholes and baby universes exist is not really a constant but fluctuates in time. This would lead to a realization of Coleman's mechanism that will be discussed in section "Coleman's Mechanism."

因此，如果不存在虫洞和婴儿宇宙的二维因果动态三角剖分 (二维 CDT) 或可投影霍拉瓦-里夫斯兹量子引力 (连续极限下) 中的宇宙学“常数”并非真正的常数，而是随时间涨落，就能完全纳入所有虫洞和婴儿宇宙求和的贡献。这会实现科尔曼机制，我们将在“科尔曼机制”一节讨论。

In the next section, we will show that the full propagator can be also obtained if quantizing 2d projectable HL gravity with an effective wormhole interaction term.

在下一节，我们将说明，当对带有效虫洞相互作用项的二维可投影霍拉瓦-里夫斯兹引力量子化时，也可以得到这个完全传播子。

Wormhole Interaction in 2d Projectable HL Gravity

二维可投影 HL 引力中的虫洞相互作用

Let us consider the 2d projectable HL gravity with a space-like wormhole interaction given by the following action:

让我们考虑存在类空虫洞相互作用的二维可投影 HL 引力，其作用量如下:

$$I_{\mathrm{w}} = \frac{1}{\kappa}\int dtdxN(t)\sqrt{h(t,x)}\left((1-\eta)K^2(t,x)-2\tilde{\Lambda}\right)$$

$$+\beta\int dtN(t)\int dx_1dx_2\sqrt{h(t,x_1)}\sqrt{h(t,x_2)}, \tag{106}$$

where β is a dimension-full coupling constant. The last bi-local term can be interpreted as an effective interaction term for a space-like wormhole connecting two distant regions at a given t . This bi-local term is allowed to be included since it is invariant under FPD.

其中 β 是一个带量纲的耦合常数。最后一项双局域项可以解释为连接给定 t 处两个遥远区域的类空虫洞的有效相互作用项。由于该项在 FPD 变换下不变，因此允许被引入。

Following essentially the same procedure explained in section "Quantization," let us quantize the system defined by the action (106). Introducing the conjugate momentum of the density $\sqrt{h}$ as Π , we introduce the Poisson bracket (72). Implementing the Legendre transform, one obtains the corresponding Hamiltonian:

遵循“量子化”小节中说明的基本相同步骤，我们对由作用量 (106) 定义的系统进行量子化。将密度 $\sqrt{h}$ 的共轭动量定义为 Π 后，我们引入泊松括号 (72)。通过勒让德变换，可得到对应的哈密顿量:

$$H_{\mathrm{w}} = N(t)\,C_{\mathrm{w}}(t) + \int dxN_1(t,x)\,C^1_{\mathrm{w}}(t,x)\,, \tag{107}$$

where

其中

$$\mathcal{C}^1_{\mathrm{w}}(t,x) = -\frac{\partial_1 \Pi(t,x)}{\sqrt{h(t,x)}} \approx 0, \tag{108}$$

$$C_{\mathrm{W}}(t) = \int dx \left(\frac{\kappa}{4(1-\eta)} \Pi^2(t,x)\sqrt{h(t,x)} + \frac{2}{\kappa}\widetilde{\Lambda}\sqrt{h(t,x)} \right.$$
$$\left. -\beta\sqrt{h(t,x)} \int dx_2 \sqrt{h(t,x_2)} \right) \approx 0. \tag{109}$$

The constraints (108) and (109) are the momentum constraint and the Hamiltonian constraint, respectively. Solving the momentum constraint (108) at the classical

约束条件 (108) 和 (109) 分别是动量约束和哈密顿量约束。和之前一样在经典

level as before, the Hamiltonian (107) reduces to the following one-dimensional one:

层面求解动量约束 (108) 后，哈密顿量 (107) 约化为如下一维形式:

$$H_{\mathrm{w}} = N(t)\left(\frac{\kappa}{4(1-\eta)} \Pi^2(t) L(t) + \frac{2}{\kappa}\widetilde{\Lambda} L(t) - \beta L^2(t) \right), \tag{110}$$

where $L(t) := \int dx \sqrt{h(t,x)}$. The Hamiltonian (110) is subject to the Hamiltonian constraint:

其中 $L(t) := \int dx \sqrt{h(t,x)}$ 。哈密顿量 (110) 满足哈密顿量约束:

$$L(t)\left(\frac{\kappa}{4(1-\eta)} \Pi^2(t) + \frac{2}{\kappa}\widetilde{\Lambda} - \beta L(t) \right) \approx 0. \tag{111}$$

Here we choose the CDT parametrization (71). A solution to the Hamiltonian constraint (111) is

此处我们选取 CDT 参数化 (71)。哈密顿量约束 (111) 的一个解是

$$\Pi^2 = -\Lambda + \beta L \geq 0, \ \ \text{for} \ \sqrt{\Lambda} L \geq 1/\xi, \tag{112}$$

where ξ is a dimensionless parameter given by $\xi = \beta/\Lambda^{3/2}$. For $\sqrt{\Lambda} L < 1/\xi$, the only allowed solution is $L = 0$.

其中 ξ 是由 $\xi = \beta/\Lambda^{3/2}$ 给出的无量纲参数。对于 $\sqrt{\Lambda} L < 1/\xi$ ，唯一允许的解是 $L = 0$ 。

When quantizing the system, if we follow the same procedure described in section ”Quantization” and set $\beta = g_s$, one can reproduce the path-integral of the full propagator (101). Remember the boundary condition for the path-integral (101), i.e., the kinetic term should counteract the unboundedness of the potential term at $L = \infty$. This balance between the kinetic and potential terms is precisely what is reflected in the classical Hamiltonian constraint (112).

对该系统量子化时，如果我们遵循“量子化”小节中描述的相同步骤并设定 $\beta = g_s$，就可以重构出完全传播子 (101) 的路径积分。请记住路径积分 (101) 的边界条件: 即动能项需要抵消势项在 $L = \infty$ 处的无界性。动能项和势项之间的这种平衡正是经典哈密顿量约束 (112) 所反映的内容。

Coleman's Mechanism

科尔曼机制

In this section, we discuss a sort of Coleman's mechanism in the context of two-dimensional gravity based on CDT briefly.

本节我们简要讨论基于 CDT 的二维引力背景下的一类科尔曼机制。

Let us define the two kinds of Wheeler-deWitt equation:

接下来我们定义两种惠勒-德维特方程:

$$\hat{H} W_0(L) = 0,\ \hat{H}_{\text{eff}} W(L) = 0, \tag{113}$$

where $\hat{H} := \hat{H}^{(-1)}$ introduced in Eq. (53). The solutions to the Wheeler-deWitt equations are the Hartle-Hawking wave functions given by

其中 $\hat{H} := \hat{H}^{(-1)}$ 已在式 (53) 中引入。惠勒-德维特方程的解即为如下形式的哈特尔-霍金波函数:

$$W_0(L) = e^{-\sqrt{\Lambda}L},\ W(L) = \frac{\text{Bi}\left(\xi^{-2/3} - \xi^{1/3}\sqrt{\Lambda}L\right)}{\text{Bi}\left(\xi^{-2/3}\right)} + c\,\text{Ai}\left(\xi^{-2/3} - \xi^{1/3}\sqrt{\Lambda}L\right),$$

(114)

where Ai and Bi are the standard Airy functions; ξ is a dimensionless string coupling constant measured by the cosmological constant, i.e., $\xi := g_s/\Lambda^{3/2}$; and c is an undetermined dimensionless constant. The Hartle-Hawking wave function $W_0(L)$ is the one for (the continuum theory of) 2 d CDT, i.e., neither baby universe nor wormhole contributions are included. On the other hand, $W(L)$ is the Hartle-Hawking wave function including all possible contributions of baby universes and wormholes non-perturbatively.

其中 Ai 和 Bi 是标准艾里函数；ξ 是由宇宙常数度量的无量纲弦耦合常数，即 $\xi := g_s/\Lambda^{3/2}$；c 是一个未定的无量纲常数。哈特尔-霍金波函数 $W_0(L)$ 对应 (连续统理论的) 2 d CDT，即不包含婴儿宇宙与虫洞贡献。另一方面，$W(L)$ 是非微扰地包含所有婴儿宇宙与虫洞贡献的哈特尔-霍金波函数。

We wish to explore the behavior of the non-perturbative Hartle-Hawking wave function $W(L)$ (see Fig.2). For $\sqrt{\Lambda}L \ll 1/\xi$, one obtains the asymptotic expansion:

我们希望探究非微扰哈特尔-霍金波函数 $W(L)$ 的行为 (参见图 2)。对于 $\sqrt{\Lambda}L \ll 1/\xi$，可得渐近展开式:

$$W(L) \sim e^{-\sqrt{\Lambda}L} = W_0(L). \tag{115}$$

Therefore, when the size of the one-dimensional universe is small enough, the physics is very closed to the one without baby universes and wormholes, and it is essentially governed by the cosmological constant. The wave function in this region decreases exponentially, and this behavior does not change at any finite order of perturbation.

因此，当一维宇宙的尺寸足够小时，其物理性质与不含婴儿宇宙和虫洞的情况非常接近，本质上由宇宙常数主导。该区域的波函数呈指数衰减，且该行为在任意有限阶微扰下都不会改变。

However, once the size of the universe is large enough, i.e., $\sqrt{\Lambda} > 1/\xi$, the wave function starts oscillating, and the behavior is governed by the string coupling constant instead of the cosmological constant:

但一旦宇宙尺寸足够大，即 $\sqrt{\Lambda} > 1/\xi$，波函数开始振荡，此时行为由弦耦合常数而非宇宙常数主导:

$$W(L) \sim 1/\left(g_s^{1/3}L\right)^{1/4}. \tag{116}$$

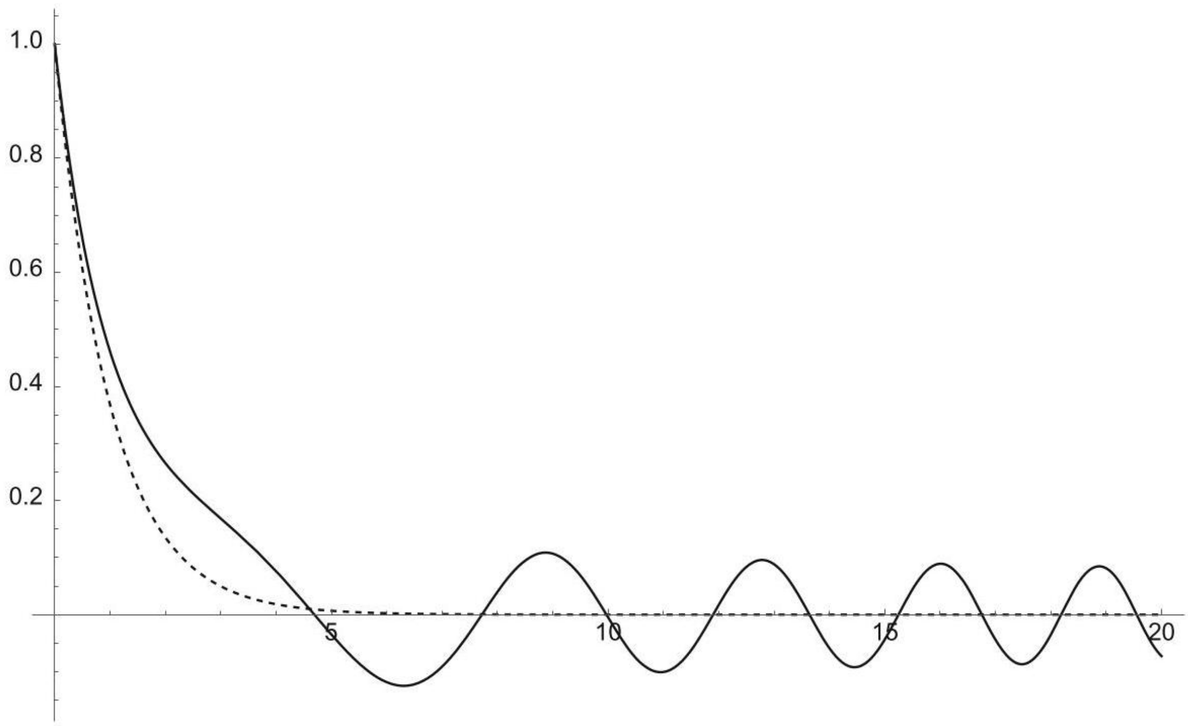


Fig. 2 A plot of the Hartle-Hawking wave functions, W_0 (dashed line) and W (solid line), for $\xi = 1/3$ and $c = 0$: The horizontal axis is $\sqrt{\Lambda}L$, and the vertical axis is either W_0 or W

图 2 当 $\xi = 1/3$ 和 $c = 0$ 时，哈特尔-霍金波函数 W_0 (虚线) 与 W (实线) 的图像: 横轴为 $\sqrt{\Lambda}L$，纵轴为 W_0 或 W

This drastic change happens due to the infinitely many wormholes and baby universes. The similar behavior has been observed in the context of non-critical string theory [34, 35] .

这种剧烈变化由无穷多虫洞与婴儿宇宙导致。类似行为曾在非临界弦理论 [34, 35] 的研究中被观测到。

From the discussion above, we observe that a sort of Coleman's mechanism works: For a large universe, the cosmological constant is not important enough to govern the physics (The Hartle-Hawking wave function $W(L)$ is not normalizable, but similar arguments are valid on the normalizable energy eigenstates [30].).

根据上述讨论，我们观测到一类科尔曼机制生效: 对于大尺度宇宙，宇宙常数不足以主导物理过程 (尽管哈特尔-霍金波函数 $W(L)$ 不可归一化，但类似论证对可归一化的能量本征态成立 [30]。)。

Summary

摘要

We have reviewed the relation between two-dimensional causal dynamical triangulations (2d CDT) and two-dimensional projectable Hořava-Lifshitz quantum gravity (2d projectable HL QG).

本文综述了二维因果动态三角剖分 (2d CDT) 与二维可投影霍拉瓦-里夫希茨量子引力 (2d 可投影 HL QG) 之间的关系。

In the first part, it has been shown that the physics described by the continuum limit of 2 d CDT coincides with the one obtained quantizing 2 d projectable HL gravity. This is confirmed because the quantum Hamiltonians of both models are exactly the same. The system is expressed in terms of quantum mechanics of a 1d extended object, i.e., a 1d universe. It would be too hasty to consider that this scenario also holds for the higher dimensional cases. In fact, it has been shown that in $2 + 1$ dimensions, numerical studies of the so-called locally causal dynamical triangulations (LCDT) that relax the proper time foliation of CDT and require the local causality reproduce an intriguing specialty of CDT, an emergence of the de-Sitter-like geometry [36] (see, e.g., Ref. [37] for the higher-dimensional CDT). On the other hand, a Landau theory approach suggests a relation between CDT and theories invariant under the foliation-preserving diffeomorphisms in $2 + 1$ dimensions [38, 39] (see also - Chap. 81, "Landau Theory of Causal Dynamical Triangulations"). This issue should be investigated further.

第一部分表明，2 d CDT 连续极限描述的物理与 2 d 可投影 HL 引力量子化得到的物理一致，这一点由两个模型的量子哈密顿量完全相同所证实。该系统可由一维延展对象即一维宇宙的量子力学描述。直接认为这一结论也适用于更高维情况过于草率。事实上已有研究表明，在 $2+1$ 维中，对放松 CDT 固有时间叶化、要求定域因果性的所谓定域因果动态三角剖分 (LCDT) 进行数值研究，重现了 CDT 一个引人关注的特性: 类德西特几何的涌现 [36](高维 CDT 相关研究可参见例如文献 [37])。另一方面，朗道理论方法表明，在 $2+1$ 维中 CDT 与保持叶化不变的微分同胚不变理论之间存在关联 [38, 39](另可参见第 81 章“因果动态三角剖分的朗道理论”)。这一问题仍有待进一步研究。

In the second part, we have introduced the generalized CDT (GCDT) that permits baby universes and wormholes to form in keeping with the foliation, and in particular the construction based on the string field theory for CDT has been explained. Here the string means the 1 d universe, and the string field theory is constructed in such a way that the free part reproduces the CDT amplitudes, and the splitting and joining interactions of string are introduced to create baby universes and wormholes.

> 第二部分，我们引入了允许婴儿宇宙与虫洞在遵守叶化规则的条件下形成的广义 CDT(GCDT)，尤其阐释了基于弦场论的 CDT 构造。此处的弦指 1 d 宇宙，弦场论的构造满足: 自由部分重现 CDT 振幅，引入弦的分裂与接合相互作用来产生婴儿宇宙和虫洞。

Focusing on the loop-to-loop amplitude, we have introduced an effective 1d theory that includes all the contributions coming from the sum over all possible baby universes and wormholes. From the point of view of HL gravity, the effective theory can be precisely reproduced if introducing a bi-local interaction term into the action of 2 d projectable HL gravity and if quantizing the system. In addition, the effective theory can be also obtained considering that the cosmological constant of 2 d CDT is not a constant, but it fluctuates in time. This leads to Coleman's mechanism in 2d CDT such that for a large universe, the cosmological constant is not important enough to govern the physics.

> 针对圈对圈振幅，我们引入了一个一维有效理论，它包含所有可能婴儿宇宙与虫洞求和得到的全部贡献。从 HL 引力的角度看，只要在 2 d 可投影 HL 引力的作用量中引入双局域相互作用项并对系统量子化，就可以精确得到该有效理论。此外，若认为 2 d CDT 的宇宙学常数并非常数，而是随时间涨落，也可以得到该有效理论，这会在二维 CDT 中催生科尔曼机制: 对大宇宙而言，宇宙学常数不足以主导物理规律。

Although we have not discussed issues of the coupling to matter, 2d CDT coupled to Yang-Mills theory has been solved analytically in Ref. [40], and it has been shown that the quantum Hamiltonian obtained in Ref. [40] can be reproduced quantizing 2d projectable HL gravity coupled to Yang-Mills theory [41]. In fact, we know very little about the analytical treatment of the coupling to matter compared to the situation of 2 d dynamical triangulations and the Liouville quantum gravity. This direction needs to be explored in the future.

> 尽管本文未讨论物质耦合问题，但已有文献 [40] 解析求解了耦合杨-米尔斯理论的二维 CDT，且结果表明，文献 [40] 得到的量子哈密顿量可以通过量子化耦合杨-米尔斯理论的二维可投影 HL 引力重现 [41]。实际上，相比于 2 d 动态三角剖分和刘维尔量子引力的研究现状，我们对物质耦合的解析处理还知之甚少，这一方向有待未来进一步探索。

What is remarkable is that following the standard Wilsonian renormalization group, one can take the continuum limit of the lattice model, 2d CDT, and find the continuum quantum field theory, 2d projectable HL QG, which is in the same universality class of 2d CDT. A missing piece is the continuum quantum field theory of GCDT that is described by metric components and allows us to compute all the amplitudes defined by the string field theory for CDT, although we have the effective field theory, the 2d projectable HL gravity with a bi-local interaction, which reproduces the restricted class of GCDT amplitudes once it is quantized. We wish to unveil the underlying continuum quantum field theory of GCDT, through which we can understand something inherently interesting about quantum geometries for sure.

值得注意的是，遵循标准威尔逊重整化群，我们可以对格点模型二维 CDT 取连续极限，得到连续量子场论——二维可投影 HL QG，它与二维 CDT 属于同一普适类。目前缺失的一块是 GCDT 的连续量子场论: 尽管我们已经得到有效场论，即带双局域相互作用的二维可投影 HL 引力，量子化后可以重现 GCDT 振幅的受限类别，但我们仍需要一个用度规分量描述、能够计算 CDT 弦场论定义的所有振幅的连续量子场论。我们希望能够揭示 GCDT 背后的连续量子场论，借此必定可以帮助我们理解量子几何中许多本质有趣的内容。

Cross-References

交叉引用

From Trees to Gravity

从树到引力

- Landau Theory of Causal Dynamical Triangulations

- 因果动力学三角剖分朗道理论

D Lessons from the Mathematics of Two-Dimensional Euclidean Quantum Gravity

二维欧几里得量子引力数学带来的 D 维经验教训

The Causality Road from Dynamical Triangulations to Quantum Gravity That Describes Our Universe

从动力学三角剖分到描述我们宇宙的量子引力的因果之路

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